Open Journal of
Discrete Applied Mathematics

PSR Press

Article

C_6-decompositions of the tensor product of complete graphs

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Received: 19 August 2020; Accepted: 15 October 2020; Published: 7 November 2020.

Abstract: Let G be a simple and finite graph. A graph is said to be decomposed into subgraphs H_1 and H_2 which is denoted by $G = H_1 \oplus H_2$, if G is the edge disjoint union of H_1 and H_2. If $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, where H_1, H_2, ..., H_k are all isomorphic to H, then G is said to be H-decomposable. Furthermore, if H is a cycle of length m then we say that G is C_m-decomposable and this can be written as $C_m|G$. Where $G \times H$ denotes the tensor product of graphs G and H, in this paper, we prove that the necessary conditions for the existence of C_6-decomposition of $K_m \times K_n$ are sufficient. Using these conditions it can be shown that every even regular complete multipartite graph G is C_6-decomposable if the number of edges of G is divisible by 6.

Keywords: Cycle decompositions, graph, tensor product.

MSC: 05C70.

1. Introduction

Let C_m, K_m and $K_m - I$ denote cycle of length m, complete graph on m vertices and complete graph on m vertices minus a 1-factor respectively. By an m-cycle we mean a cycle of length m. All graphs considered in this paper are simple and finite. A graph is said to be *decomposed* into subgraphs H_1 and H_2 which is denoted by $G = H_1 \oplus H_2$, if G is the edge disjoint union of H_1 and H_2. If $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, where H_1, H_2, ..., H_k are all isomorphic to H, then G is said to be H-decomposable. Furthermore, if H is a cycle of length m then we say that G is C_m-decomposable and this can be written as $C_m|G$. A k-factor of G is a k-regular spanning subgraph. A k-factorization of a graph G is a partition of the edge set of G into k-factors. A C_k-factor of a graph is a 2-factor in which each component is a cycle of length k. A *resolvable* k-cycle decomposition (for short k-RCD) of G denoted by $C_k||G$, is a 2-factorization of G in which each 2-factor is a C_k-factor.

For two graphs G and H their tensor product $G \times H$ has vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$. From this, note that the tensor product of graphs is distributive over edge disjoint union of graphs, that is if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, then $G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \cdots \oplus (H_k \times H)$. Now, for $h \in V(H)$, $V(G) \times h = \{(v, h) | v \in V(G)\}$ is called a column of vertices of $G \times H$ corresponding to h. Further, for $y \in V(G)$, $y \times V(H) = \{(y, v) | v \in V(H)\}$ is called a layer of vertices of $G \times H$ corresponding to y.

In [1], Oyewumi *et al.*, obtained an interesting result on the decompositions of certain graphs. The problem of finding C_k-decomposition of K_{2n+1} or $K_{2n} - I$ where I is a 1-factor of K_{2n}, is completely settled by Alspach, Gavlas and Sajna in two different papers (see [2,3]). A generalization to the above complete graph decomposition problem is to find a C_k-decomposition of $K_m * \overline{K}_n$, which is the complete m-partite graph in which each partite set has n vertices. The study of cycle decompositions of $K_m * \overline{K}_n$ was initiated by Hoffman *et al.*, [4]. In the case when p is a prime, the necessary and sufficient conditions for the existence of C_p-decomposition of $K_m * \overline{K}_n$, $p \geq 5$ is obtained by Manikandan and Paulraja in [5–7]. Billington [8] studied the decomposition of complete tripartite graphs into cycles of length 3 and 4. Furthermore, Cavenagh and Billington [9] studied 4-cycle, 6-cycle and 8-cycle decomposition of complete multipartite graphs.

Billington *et al.*, [10] solved the problem of decomposing $(K_m * \overline{K}_n)$ into 5-cycles. Similarly, when $p \geq 3$ is a prime, the necessary and sufficient conditions for the existence of C_{2p}-decomposition of $K_m * \overline{K}_n$ was obtained

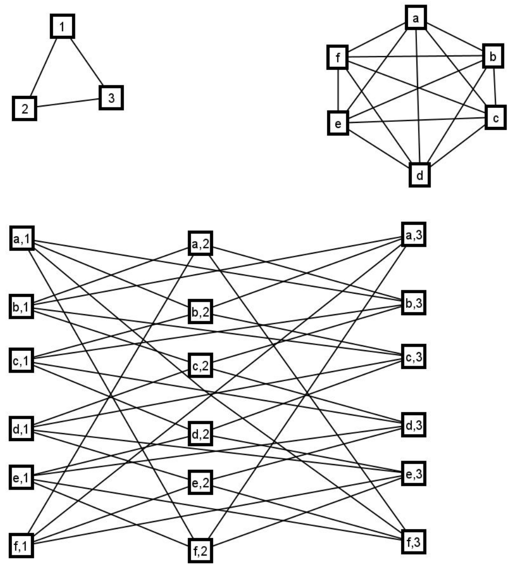

Figure 1. The tensor product $C_3 \times K_6$. C_3 and K_6 are shown at the top of the product respectively.

by Smith (see [11]). For a prime $p \geq 3$, it was proved in [12] that C_{3p}-decomposition of $K_m * \overline{K}_n$ exists if the obvious necessary conditions are satisfied. As the graph $K_m \times K_n \cong K_m * \overline{K}_n - E(nK_m)$ is a proper regular spanning subgraph of $K_m * \overline{K}_n$. It is natural to think about the cycle decomposition of $K_m \times K_n$.

The results in [5–7] also give necessary and sufficient conditions for the existence of a p-cycle decomposition, (where $p \geq 5$ is a prime number) of the graph $K_m \times K_n$. In [13] it was shown that the tensor product of two regular complete multipartite graph is Hamilton cycle decomposable. Muthusamy and Paulraja in [14] proved the existence of C_{kn}-factorization of the graph $C_k \times K_{mn}$, where $mn \neq 2 (\text{mod } 4)$ and k is odd. While Paulraja and Kumar [15] showed that the necessary conditions for the existence of a resolvable k-cycle decomposition of tensor product of complete graphs are sufficient when k is even. Oyewumi and Akwu [16] proved that C_4 decomposes the product $K_m \times K_n$, if and only if either (1) $n \equiv 0 \pmod 4$ and m is odd, (2) $m \equiv 0 \pmod 4$ and n is odd or (3) m *or* $n \equiv 1 \pmod 4$.

As a companion to the work in [16], i.e., to consider the decomposition of the tensor product of complete graphs into cycles of even length. This paper proves that the obvious necessary conditions for $K_m \times K_n$, $2 \leq m, n$, to have a C_6-decomposition are also sufficient. Among other results, here we prove the following main results.

It is not surprising that the conditions in Theorem 1 are "symmetric" with respect to m and n since $K_m \times K_n \cong K_n \times K_m$.

Theorem 1. *Let $2 \leq m, n$, then $C_6 | K_m \times K_n$ if and only if $m \equiv 1$ or 3 (mod 6) or $n \equiv 1$ or 3 (mod 6).*

Theorem 2. *Let m be an even integer and $m \geq 6$, then $C_6 | K_m - I \times K_n$ if and only if $m \equiv 0$ or 2 (mod 6).*

2. C_6 decomposition of $C_3 \times K_n$

Theorem 3. *Let $n \in N$, then $C_6 | C_3 \times K_n$.*

Proof. Following from the definition of tensor product of graphs, let $U^1 = \{u_1, v_1, w_1\}$, $U^2 = \{u_2, v_2, w_2\}$,..., $U^n = \{u_n, v_n, w_n\}$ form the partite set of vertices in $C_3 \times K_n$. Also, U^i and U^j has an edge in $C_3 \times K_n$ for $1 \leq i, j \leq n$ and $i \neq j$ if the subgraph induce $K_{3,3} - I$, where I is a 1-factor of $K_{3,3}$. Now, each subgraph $U^i \cup U^j$ is isomorphic to $K_{3,3} - I$. But $K_{3,3} - I$ is a cycle of length six. Hence the proof. □

Example 1. The graph $C_3 \times K_7$ can be decomposed into cycles of length 6.

Proof. Let the partite sets (layers) of the tripartite graph $C_3 \times K_7$ be $U = \{u_1, u_2, ..., u_7\}$, $V = \{v_1, v_2, ..., v_7\}$ and $W = \{w_1, w_2, ..., w_7\}$. We assume that the vertices of U, V and W having same subscripts are the corresponding

vertices of the partite sets. A 6-cycle decomposition of $C_3 \times K_7$ is given below:
$\{u_1, v_2, w_1, u_2, v_1, w_2\}$,$\{u_1, v_3, w_1, u_3, v_1, w_3\}$,$\{u_2, v_3, w_2, u_3, v_2, w_3\}$,
$\{u_1, v_4, w_1, u_4, v_1, w_4\}$,$\{u_2, v_4, w_2, u_4, v_2, w_4\}$,$\{u_3, v_4, w_3, u_4, v_3, w_4\}$,
$\{u_1, v_5, w_1, u_5, v_1, w_5\}$,$\{u_2, v_5, w_2, u_5, v_2, w_5\}$,$\{u_3, v_5, w_3, u_5, v_3, w_5\}$,
$\{u_4, v_5, w_4, u_5, v_4, w_5\}$,$\{u_1, v_6, w_1, u_6, v_1, w_6\}$,$\{u_2, v_6, w_2, u_6, v_2, w_6\}$,
$\{u_3, v_6, w_3, u_6, v_3, w_6\}$,$\{u_4, v_6, w_4, u_6, v_4, w_6\}$,$\{u_5, v_6, w_5, u_6, v_5, w_6\}$,
$\{u_1, v_7, w_1, u_7, v_1, w_7\}$,$\{u_2, v_7, w_2, u_7, v_2, w_7\}$,$\{u_3, v_7, w_3, u_7, v_3, w_7\}$,
$\{u_4, v_7, w_4, u_7, v_4, w_7\}$,$\{u_5, v_7, w_5, u_7, v_5, w_7\}$,$\{u_6, v_7, w_6, u_7, v_6, w_7\}$. □

Theorem 4. *[17] Let m be an odd integer and $m \geq 3$. If $m \equiv 1$ or $3 \pmod 6$ then $C_3 | K_m$.*

Theorem 5. *[3] Let n be an even integer and m be an odd integer with $3 \leq m \leq n$. The graph $K_n - I$ can be decomposed into cycles of length m whenever m divides the number of edges in $K_n - I$.*

3. C_6 decomposition of $C_6 \times K_n$

Theorem 6. *[3] Let n be an odd integer and m be an even integer with $3 \leq m \leq n$. The graph K_n can be decomposed into cycles of length m whenever m divides the number of edges in K_n.*

Lemma 1. $C_6 | C_6 \times K_2$.

Proof. Let the partite set of the bipartite graph $C_6 \times K_2$ be $\{u_1, u_2, ..., u_6\}$, $\{v_1, v_2, ..., v_6\}$. We assume that the vertices having the same subscripts are the corresponding vertices of the partite sets. Now $C_6 \times K_2$ can be decomposed into 6-cycles which are $\{u_1, v_2, u_3, v_4, u_5, v_6\}$ and $\{v_1, u_2, v_3, u_4, v_5, u_6\}$. □

Theorem 7. *For all n, $C_6 | C_6 \times K_n$.*

Proof. Let the partite set of the 6-partite graph $C_6 \times K_n$ be $U = \{u_1, u_2, ..., u_n\}$, $V = \{v_1, v_2, ..., v_n\}$, $W = \{w_1, w_2, ..., w_n\}$, $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_n\}$ and $Z = \{z_1, z_2, ..., z_n\}$, we assume that the vertices of U, V, W, X, Y and Z having the same subscripts are the corresponding vertices of the partite sets. Let $U^1 = \{u_1, v_1, w_1, x_1, y_1, z_1\}$, $U^2 = \{u_2, v_2, w_2, x_2, y_2, z_2\}$, ..., $U^n = \{u_n, v_n, w_n, x_n, y_n, z_n\}$ be the sets of these vertices having the same subscripts. By the definition of the tensor product, each U^i, $1 \leq i \leq n$ is an independent set and the subgraph induced by each $U^i \cup U^j$, $1 \leq i, j \leq n$ and $i \neq j$ is isomorphic to $C_6 \times K_2$. Now by Lemma 1 the graph $C_6 \times K_2$ admits a 6-cycle decomposition. This completes the proof. □

4. C_6 decomposition of $K_m \times K_n$[Proofs of main Theorems]

Proof of Theorem 1. Assume that $C_6 | K_m \times K_n$ for some m and n with $2 \leq m, n$. Then every vertex of $K_m \times K_n$ has even degree and 6 divides in the number of edges of $K_m \times K_n$. These two conditions translate to $(m-1)(n-1)$ being even and $6 | m(m-1)n(n-1)$ respectively. Hence, by the first fact m or n has to be odd, i.e., has to be congruent to 1 *or* 3 *or* 5 (mod 6). The second fact can now be used to show that they cannot both be congruent to 5 (mod 6). It now follows that $m \equiv 1$ *or* $3 \pmod 6$ or $n \equiv 1$ *or* $3 \pmod 6$.

Conversely, let $m \equiv 1$ *or* $3 \pmod 6$. By Theorem 4, $C_3 | K_m$ and hence $K_m \times K_n = ((C_3 \times K_n) \oplus \cdots \oplus (C_3 \times K_n))$. Since $C_6 | C_3 \times K_n$ by Theorem 3.

Finally, if $n \equiv 1$ *or* $3 \pmod 6$, the above argument can be repeated with the roles of m and n interchanged to show again that $C_6 | K_m \times K_n$. This completes the proof. □

Proof of Theorem 2. Assume that $C_6 | K_m - I \times K_n, m \geq 6$. Certainly, $6 | mn(m-2)(n-1)$. But we know that if $6 | m(m-2)$ then $6 | mn(m-2)(n-1)$. But m is even therefore $m \equiv 0$ *or* $2 \pmod 6$.

Conversely, let $m \equiv 0$ *or* $2 \pmod 6$. Notice that for each m, $\frac{m(m-2)}{2}$ is a multiple of 3. Thus by Theorem 5 $C_3 | K_m - I$ and hence $K_m - I \times K_n = ((C_3 \times K_n) \oplus \cdots \oplus (C_3 \times K_n))$. From Theorem 3, $C_6 | C_3 \times K_n$. The proof is complete. □

5. Conclusion

In view of the results obtained in this paper we draw our conclusion by the following corollary.

Corollary 1. *For any simple graph G. If*

1. $C_3|G$ *then* $C_6|G \times K_n$*, whenever* $n \geq 2$.
2. $C_6|G$ *then* $C_6|G \times K_n$*, whenever* $n \geq 2$.

Proof. We only need to show that $C_3|G$. Applying Theorem 3 gives the result. □

Acknowledgments: The authors would like to thank the referees for helpful suggestions which has improved the present form of this work.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

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