British Journal of Mathematics & Computer Science
20(4): 1-17, 2017; Article no.BJMCS.31191
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

Calculation of Banzhaf Voting Indices Utilizing Variable-Entered Karnaugh Maps

Ali Muhammad Ali Rushdi[1*] and Omar Mohammed Ba-Rukab[2]

[1]Department of Electrical and Computer Engineering, Faculty of Engineering, King Abdulaziz University, P.O.Box 80204, Jeddah 21589, Saudi Arabia.
[2]Department of Information Technology, Faculty of Computing and Information Technology, King Abdulaziz University, P.O.Box 344, Rabigh 21911, Saudi Arabia.

Authors' contributions

The two authors collaborated to accomplish this research work. Author AMAR envisioned and designed this study, drafted the manuscript and surveyed the pertinent literature. Author OMBR managed the analysis, implemented the algorithm, drew the figures and contributed to literature survey. Both authors proofread and finalized the manuscript.

Article Information

DOI: 10.9734/BJMCS/2017/31191
Editor(s):
(1) Serkan Araci, Hasan Kalyoncu University, Turkey.
Reviewers:
(1) Xingting Wang, Temple University, USA.
(2) Ajay Auddy, The University of Burdwan, West Bengal, India.
(3) Ruffin-Benoît M. Ngoie, Institut Supérieur Pédagogique de Mbanza-Ngungu, RD Congo.
Complete Peer review History: http://www.sciencedomain.org/review-history/17676

Original Research Article

Received: 25th December 2016
Accepted: 20th January 2017
Published: 30th January 2017

Abstract

This paper is a tutorial exposition on how to translate concepts of voting systems to the Boolean domain, and consequently on how to use Boolean tools in the computation of a prominent index of voting powers, *viz.*, the Banzhaf voting index. We discuss Boolean representations for yes-no voting systems, in general, and for weighted voting systems, in particular. Our main observation is that non-minimal winning coalitions are related to minimal ones *via* partial-order structures and also as particular subordinate loops that cover the all-1 cell in the Karnaugh map. We review the method of computing the total Banzhaf indices by the Conventional Karnaugh Map (CKM). Then we extend this method to handle larger problems *via* the Variable-Entered Karnaugh Map (VEKM). The map methods are demonstrated by two classical weighted voting systems.

**Corresponding author: E-mail: arushdi@kau.edu.sa;*
Co-author: E-mail: obarukab@kau.edu.sa;

Keywords: *Voting system; Banzhaf index; coalition; Conventional Karnaugh map; Variable-Entered Karnaugh Map.*

1 Introduction

A weighted voting system [1] is a special case of yes-no voting system, wherein a weight is assigned to each voter and an appropriate decision threshold is fixed. A resolution (proposal) is passed (accepted) if the sum of weighted votes in favor of it reaches or exceeds the given threshold. A prominent measure of the voting power of an individual voter P in such a system is called the Total Banzhaf IndexTBI(P) of such a voter [1,2,3,4,5]. It is the number of winning states or configurations in which the voter is among supporters or proponents of the resolution (proposal) such that a switch of the voter to join the opponents changes the state from winning to losing. Rushdi & Ba-Rukab [5] named each of the 2^n states or configurations of an n-member weighted voting system a primitive coalition. The Total Banzhaf Index TBI(P) is then the number of winning primitive coalitions (WPCs) of which voter P is a member such that defection of P causes a transition to a losing primitive coalition (LPC). Implicit in the definition of the Banzhaf index is the assumption that system states or configurations (primitive coalitions) are equally likely. This in turn necessitates that voters cast their votes independently of each other.

The aim of this paper is to explore map methods for the calculation of Banzhaf voting powers or indices. The methods are illustrated via the prominent examples of the European Economic Community (EEC), a 6-member weighted voting system and the Extended European Economic Community (EEEC), a 9-member weighted voting system, which are two predecessors of the European Union (EU) of today. The paper reviews the use of the conventional Karnaugh Map (KCM) for computing TBI(P) for systems with up to 6 members and then utilizes the Variable-Entered Karnaugh Map (VEKM) systems with up to 12 members.

The organization of the rest of the paper is as follows. Section 2 explores the Boolean description of a yes-no voting system. Section 3 classifies mathematical representations of a weighted voting system. Section 4 reviews the use of a CKM in computing Banzhaf indices, while section 5 extends the map utility for that task by using the more powerful VEKM representation. Section 6 concludes the paper.

2 Boolean Representation of a Yes-no Voting System

The passage of a resolution in a yes-no voting system might be described by a switching function, i.e., a two-valued Boolean function, henceforth called simply a Boolean function) that is 1 if the resolution is passed and 0 if it is rejected. This Boolean function will be called herein the system success function and will be denoted by g(X). Here the binary vector $X = [X_1 X_2 \dots X_n]^T$ is an n-tuple of the votes X_i [$1 \leq i \leq n$] cast by voters, where X_i is 1 or 0 if voter i says 'yes' or 'no', respectively. Table 1 indicates that prominent properties that might be enjoyed by a yes-no voting system are inherited by its success Boolean function g(X). Note that the property of causality is implied by that of monotonicity. A yes-no voting system which possesses these two properties is called monotone, while a success function g(X)possessing them is called monotonically non-decreasing (or simply monotonically increasing) and sometimes called semi coherent. The function is (fully) coherent if, in addition, it has the relevancy property.

If a coherent function g(X) is represented on a CKM, the three properties of Table 1 are manifested as [6,7]:

1. Causality: The CKM is entered by 0 in the all-0 cell (the cell $\overline{X}_1 \overline{X}_2 \dots \overline{X}_n$ within the non-asserted domain of every variable), and entered by 1 in the all-1 cell (the cell $X_1 X_2 \dots X_n$ within the asserted domain of every variable).
2. Monotonicity: The map entry cannot decrease (from 1 to 0) upon transition from the non-asserted half-map domain of a variable X_i to its asserted half-map domain, with states of other variables unchanged.

3. Relevancy: For every i ($1 \le i \le n$), there exists at least one instance in which crossing the border of variable X_i from its non-asserted half-map domain to its asserted half-map domain leads to an increase of cell entry (from 0 to 1).

Such a coherent function g(X) is a unate function [8], and can have a sum-of-products (sop) representation consisting solely of uncomplemented literals. Examples of such a representation are the function's minimal sum (minimal sop covering it) and its complete sum (disjunction of all prime implicants) which are the same [9]. Since the complete sum of any Boolean function is unique and canonical, this means that the minimal sum of a coherent g(X) is also unique and canonical.

3 Mathematical Representations of a Weighted Voting System

This section surveys various mathematical representations of a weighted voting system. These include listing of its weights and threshold, use of inequalities involving a pseudo-Boolean function (expressed in algebraic or map form), utilization of a threshold Boolean function (again, expressed in algebraic or map form), listing of all winning coalitions (WCs) or, in particular, listing of all minimal winning coalitions (MWCs).

3.1 Listing of weights and threshold

An n-member weighted voting system is usually described by an n-dimensional vector of weights

$$W = [W_1 W_2 \dots W_n]^T$$

and a threshold T. In the case o the original European Economic Community (EEC), which lasted from 1958 to 1973 [1], these are given by

$$W = [W_F \quad W_G \quad W_I \quad W_B \quad W_N \quad W_L]^T = [\,4 \quad 4 \quad 4 \quad 2 \quad 2 \quad 1\,]^T, \tag{1a}$$

$$T = 12. \tag{1b}$$

Here the subscripts F, G, I, B, N, and L stand for France, Germany, Italy, Belgium, the Netherland and Luxemburg, respectively. Note that the threshold constitutes (12/17)= 70.6% of the total weights $\sum_{i=1}^{6} W_i$. In 1973, the first enlargement of the EEC took place making it the Extended European Economic Community (EEEC) by adding three new members, namely: Britain (R), Denmark (D) and Ireland (E). The weight vector and threshold became:

$$\begin{aligned} W' &= [W'_F \quad W'_G \quad W'_I \quad W'_R \quad W'_B \quad W'_N \quad W'_D \quad W'_E \quad W'_L]^T \\ &= [10 \quad 10 \quad 10 \quad 10 \quad 5 \quad 5 \quad 3 \quad 3 \quad 2\,]^T \end{aligned} \tag{2a}$$

and

$$T' = 41. \tag{2b}$$

The new threshold is now equal to (41/58)=70.7% of the new sum of weights $\sum_{i=1}^{9} W_i{}'$.

3.2 Inequalities involving a Pseudo-Boolean function

A weighted voting system can be represented by an inequality of the form

$$G(X) = W^T X = \sum_{i=1}^{n} W_i X_i \ge T. \tag{3}$$

Here the function $G(X)$ is a pseudo-Boolean ($G(X): B_2^n \rightarrow R$) where B_2 is the Boolean carrier {0, 1}, R is the real line, and the vector $X = [X_1 X_2 \ldots X_n]^T$ is an n-dimensional vector of binary indicators X_i ($1 \leq i \leq n$) for voter i, such that $X_i = 1$ if voter i supports the decision, and $X_i = 0$ if voter i opposes the decision. The representation (3) for the EEC and EEEC is, respectively

$$\begin{aligned} G(X) &= W_F F + W_G G + W_I I + W_B B + W_N N + W_L L \\ &= 4F + 4G + 4I + 2B + 2N + L \geq 17. \end{aligned} \tag{4}$$

and

$$\begin{aligned} G'(X) &= W'_F F + W'_G G + W'_I I + W'_R R + W'_B B + W'_N N + W'_D D + W'_E E + W'_L L \\ &= 10F + 10G + 10I + 10R + 5B + 5N + 3D + 3E + 2L \geq 41. \end{aligned} \tag{5}$$

Beside the algebraic forms above for $G(X)$ and $G'(X)$, these pseudo-Boolean functions have convenient CKM or VEKM representations as we will see shortly.

3.3 Threshold Boolean functions

A weighted voting system can be represented by a threshold Boolean function $g(X)$ such that [7,10,11]

$$\{g(X) = 1\} \Leftrightarrow \{G(X) = \textstyle\sum_{i=1}^{n} W_i X_i \geq T\}. \tag{6}$$

In the sequel, we will show how to obtain map representations for $g_2(X)$ of the EEC system and $g'_2(X)$ for the EEEC system.

3.4 Winning coalitions

A weighted voting system is also uniquely characterized by the set of winning coalitions. A winning coalition is any set of voters whose siding by a proposal secures its approval. For a coherent or semi-coherent system, the grand coalition (set of all voters) is a winning coalition. For example, the grand coalition {F, G, I, B, N, L} of the EEC system is a winning coalition. A common practice in the literature (see, *e.g*., Taylor & Pacelli [1]) is to abandon the parenthetical notation of sets to a form of juxtapositioning in which the above grand coalition is written simply as FGIBNL. We follow this little abuse of notation in listing all winning coalitions of the EEC system in Fig. 1. This figure has a totally ordered set of horizontal lines depicting coalition weights, with a minimal value of 12 corresponding to the threshold and a maximum value of 17 expressing the sum of voter weights or the weight of the grand coalition. Set containment (partial order) relations are depicted by directed non-horizontal lines such that a superset is at the upper end of a straight line segment with a corresponding immediate subset at its lower end. The arrow of each directed line is associated with the name of single voter that is to be added to a subset to obtain the corresponding immediate superset. Clearly the grand coalition is a superset of any winning coalition. Each of the four winning coalitions FGI, FGBN, FIBN, and GIBN is not a subset of any other coalition, and is a minimal winning coalition (MWC). Clearly, the MWCs can be used to generate all winning coalitions and hence they also suffice to uniquely characterize the voting system.

In passing, we note that a conventional winning coalition (which is a set of 'yes' voters) will be seen shortly to correspond to an implicant without complemented literals of the system success function $g(X)$. By contrast, a winning primitive coalition other than the grand coalition is a set of both, 'yes' voters and 'no' voters and corresponds to a minterm implicant of $g(X)$ that possesses both complemented and uncomplemented literals. The terminology winning primitive coalition (WPC) is not within the standard jargon of the topic, but together with its complementary term "losing primitive coalition (LPC)", it facilitates discussions concerning specific system states or configurations. The two terms provide direct counterparts to the Boolean terms of a true minterm (true vector) and false minterm (false vector).

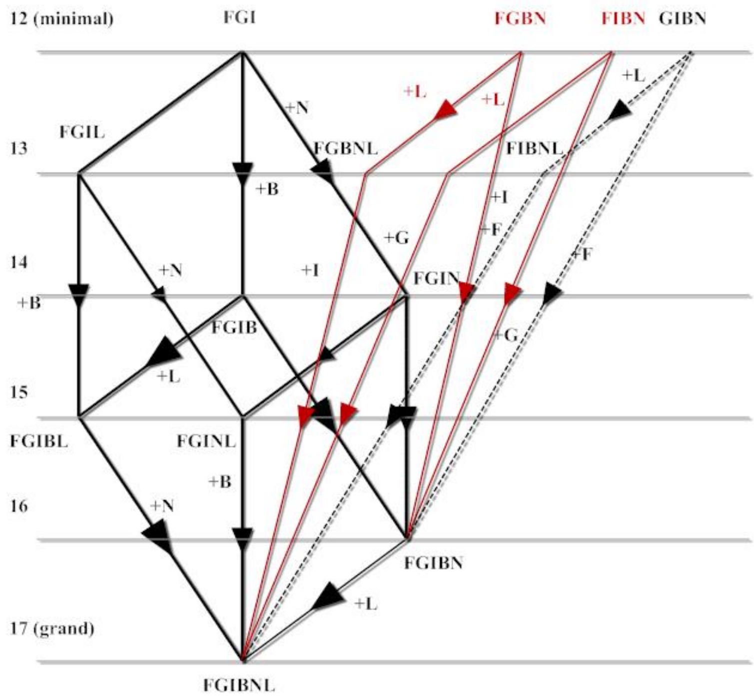


Fig. 1. A Partial-order structure depicting relations among minimal and non-minimal winning coalitions. This structure is superposed on a totally ordered set of horizontal lines depicting coalition weights. The grand coalition (FGIBNL) subsumes (is a superset of) every winning coalition

4 Computation of the Banzhaf Indices

This section explains the meaning and computation of the Banzhaf indices [1,2,12] *via* Karnaugh map representations of pseudo-Boolean and threshold Boolean functions. The section sets the stage for our novel contribution in which we compute these indices *via* the Variable-Entered Karnaugh Map rather than the Conventional Karnaugh Map. Now, we demonstrate how to compute the Banzhaf power indices for the EEC system. We draw in Fig. 2 a six-variable Karnaugh map to represent the pseudo-switching function represented in the L.H.S. of Equation (4). Each map cell represents either a winning primitive coalition (WPC) when its entry equals or exceeds the threshold of 12, or a losing primitive coalition (LPC) when its entry is less than 12. We distinguish a WPC by highlighting its weight (cell entry) in bold, while keeping that of a LPC at light color. Now, to obtain the total Banzhaf index of France TBI(F), we note that the map can be split into two halves, namely (a) the asserted domain of F where F=1 (the right half of the map, shown shaded), and (b) the non-asserted domain of F where F=0 (the left half of the map, shown unshaded). The border between these two F-domains is an F-axis of symmetry that bisects the physical normal distance between two cells in a pair of mirror-image cells *w.r.t.* the F-border. Any of these pairs of mirror-image cells has its two cells separated by a unit Hamming distance since they differ only in the value of variable F while sharing the same value of other variables.

Now, we note that coherency dictates that a pair of mirror-image cells cannot be a LPC cell in the asserted domain with a corresponding WPC cell in the non-asserted domain. There are pairs of mirror-image cells of the same type of cell in the asserted and non-asserted domain (WPC-WPC pairs or LPC-LPC pairs), but we are not interested in them. What we need to identify is the pairs of mirror-image cells with a WPC in the asserted domain and a LPC in the non asserted domain. Fig. 2 does exactly this task by using arrows to indicate transitions across the F-border from a WPC cell (with its bold entry enclosed in a continuous ellipse), to a mirror-image LPC cell (with its non-bold entry encircled in a dotted ellipse). There are 10 such transitions in Fig. 2, which are each from a winning primitive coalition of which France (F) is a member such that her defection renders the coalition a losing one. Therefore, the number of these transitions expresses TBI(F) and hence

$$\mathrm{TBI(F)} = 10. \tag{7a}$$

The job done for F by Fig. 2 is repeated by Figs. 3 and 4 for B and L, respectively, and lead to

$$\mathrm{TBI(B)} = 6. \tag{7b}$$

$$\mathrm{TBI(L)} = 0. \tag{7c}$$

This job is not repeated for G, I, or N since we know by symmetry that

$$\mathrm{TBI(G)} = \mathrm{TBI(I)} = \mathrm{TBI(F)}, \tag{7d}$$

$$\mathrm{TBI(N)} = \mathrm{TBI(B)}. \tag{7e}$$

Table 2 summarizes our results for the total Banzhaf powers, which we call 'raw' values. Table 2 shows also two kinds of normalizations of these values. The first normalization entails dividing the raw values by the total number of configurations (the number of cells in the Karnaugh maps), which is $2^{\mathrm{n}} = 2^6 = 64$. An alternative, and more appealing, normalization is to divide each of the raw values by their sum, thereby allowing interpreting the resulting normalized indices as probabilities, and allowing easy comparison with the other prominent type of indices, such as the Shapley-Shubik indices [13].

Now, we repeat our work using the Karnaugh map representation of the success function of the EEC system in Fig. 5. This function is the threshold Boolean function $\mathrm{g}_1(\mathrm{X})$ that is 1 iff $\mathrm{G}_1(\mathrm{X})$ in either of Figs. 2-4 is equal to or exceeds 12. The Karnaugh map in Fig. 5 is the same as those in Figs. 2-4 with a bold entry replaced by 1 and a non-bold entry replaced by 0. In Fig. 6, we obtain the Boolean difference (derivative) [7, 11,14,15].

$$\frac{\partial \mathrm{g}_1}{\partial \mathrm{F}} = \mathrm{g}_1(\mathrm{F, G, I, B, N, L}) \oplus \mathrm{g}_1(\overline{\mathrm{F}}, \mathrm{G, I, B, N, L}) = \mathrm{g}_1(1, \mathrm{G, I, B, N, L}) \oplus \mathrm{g}_1(0, \mathrm{G, I, B, N, L}). \tag{8}$$

by combining every two cells in a pair of cells that are mirror-images *w.r.t.* the F-border into a single cell. The entry of the combined cell is obtained by XORing the entries of its parent cells according to the rules

$$1 \oplus 0 = 1. \tag{8a}$$

$$0 \oplus 0 = 1 \oplus 1 = 0. \tag{8b}$$

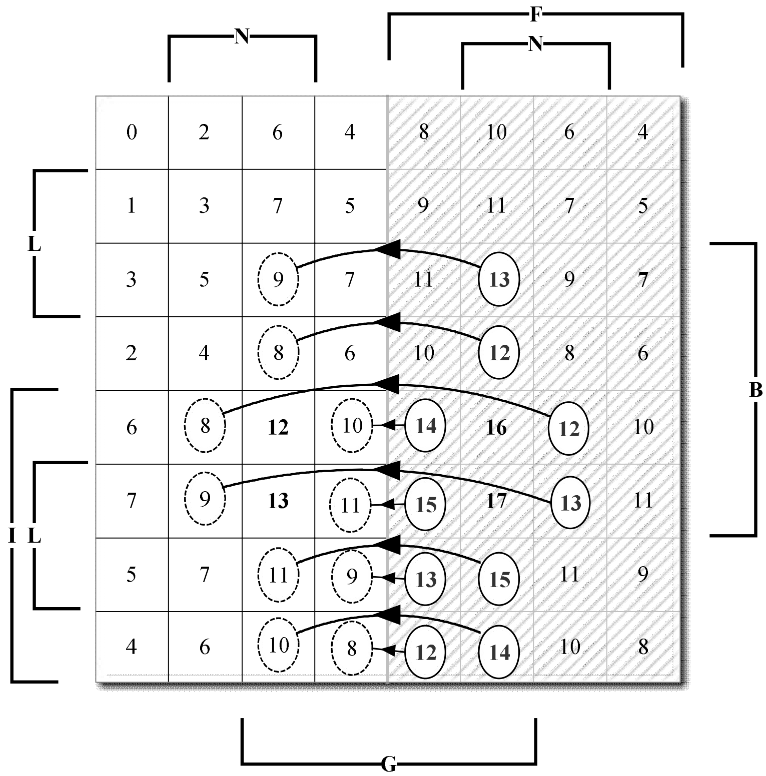
Now, TBI(F) is the number of true vectors (1-entred cells) in Fig. 6, viz. 10. Similarly, Fig. 7 indicates that TBI(B) = 6. We deliberately omit a similar figure for computing TBP(L), since it obviously lacks any true cells and represents the Boolean constant 0, thereby indicating that TBI(L) =0.

To close this section, we revisit the concept of minimal coalitions (earlier discussed *via* Fig. 1) from a Boolean perspective. We cover the function $\mathrm{g}_1(\mathrm{X})$ in Fig. 5 by its minimal sum (minimal sum-of-products) representation, using four continuous prime-implicant loops, viz;

$$\mathrm{g}_1(\mathrm{X}) = \mathrm{FGI} \vee \mathrm{FGBN} \vee \mathrm{FIBN} \vee \mathrm{GIBN}. \tag{9}$$

Despite the striking similarity of Equation (9) with the top line in Fig. 1 containing all minimal winning coalitions (MWCs), we stress again that the same notation is used for different entities. For example, FGI in

Equation (9) is a Boolean product or term $(F \wedge G \wedge I)$ of indicator variables F, G, and I that is a prime implicant of the Boolean function $g_1(X)$, while FGI in Fig. 1 (better understood as {F, G, I}) is a set of members F, G, and I that are mere symbols of the respective countries. Each of the four terms in Equation (9) is a prime implicant since its loop cannot be contained in a larger loop. Each of them is essential since it is the only prime-implicant loop to cover particular 1-entred cells (shown starred in Fig. 8). Fig. 8 does not only illustrate the minimal winning coalitions as prime implicants of the success function, but it is useful in identifying other non-minimal winning coalitions as well. An example of these non-minimal winning coalitions is FGIL, represented by a dotted loop in Fig. 8. In general, a non-minimal winning coalition is represented by a product of non-complemented literals, *i.e.*, one that is subsumed by the product of the grand coalition, and hence its loop must cover the all-1 cell. Note that the 8-cell loop FGI has a product that is subsumed by the 8 products FGI, FGIB, FGIN, FGIL, FGIBN, FGIBL, FGINL, and FGIBNL. This subsumption relation corresponds to the cubical lattice in Fig. 1 depicting partial order with top element FGI and Bottom element FGIBNL. Similarly, each of the 4-cell loops FGBN, FIBN, and GIBN has a product that subsumes 4 products corresponding to a sub-cubical lattice in Fig. 1.

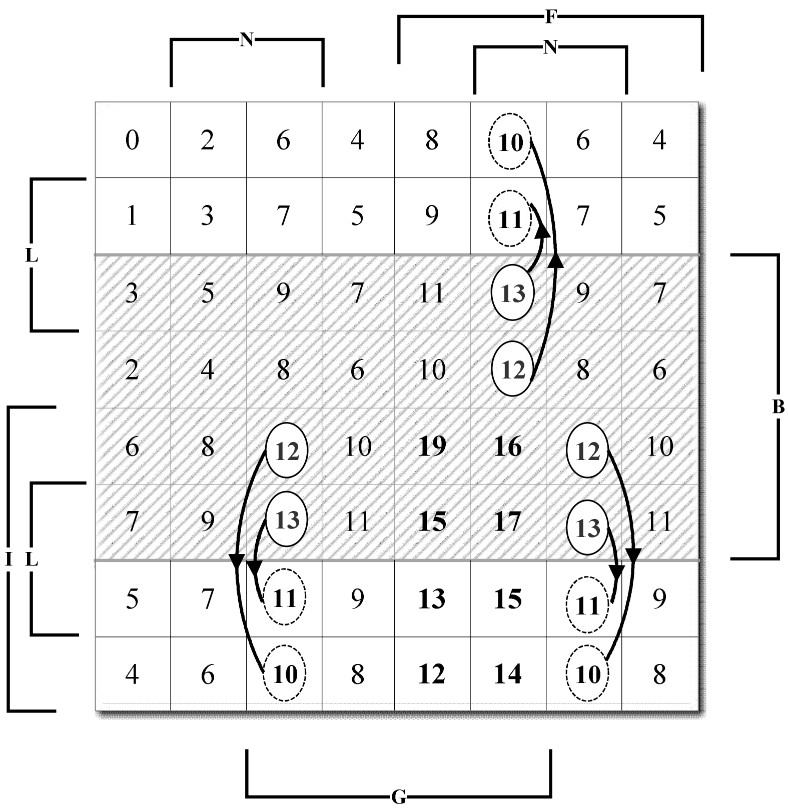


$$G_1 = 4F + 4G + 4I + 2B + 2N + L.$$

Fig. 2. A six-variable Karnaugh map for the pseudo-switching function G_1 in the L.H.S. of (4). Bold entries are those greater than or equal to the threshold(12) and represent winning primitive coalitions (WPCs). The asserted region of F is shaded. The weight of a WPC of which France (F) is a member is encircled in a continuous ellipse when defection of France is critical, *i.e.*, when transition across the F-boundary leads to a losing primitive coalition (LPC), *i.e.,* to a configuration of a weight less than the threshold (non-bold) encircled in a dotted ellipse. The figure indicates that TBI(F) is the number of arrows (transitions from a WPC to a mirror image WLC across the F boundary) which is 10

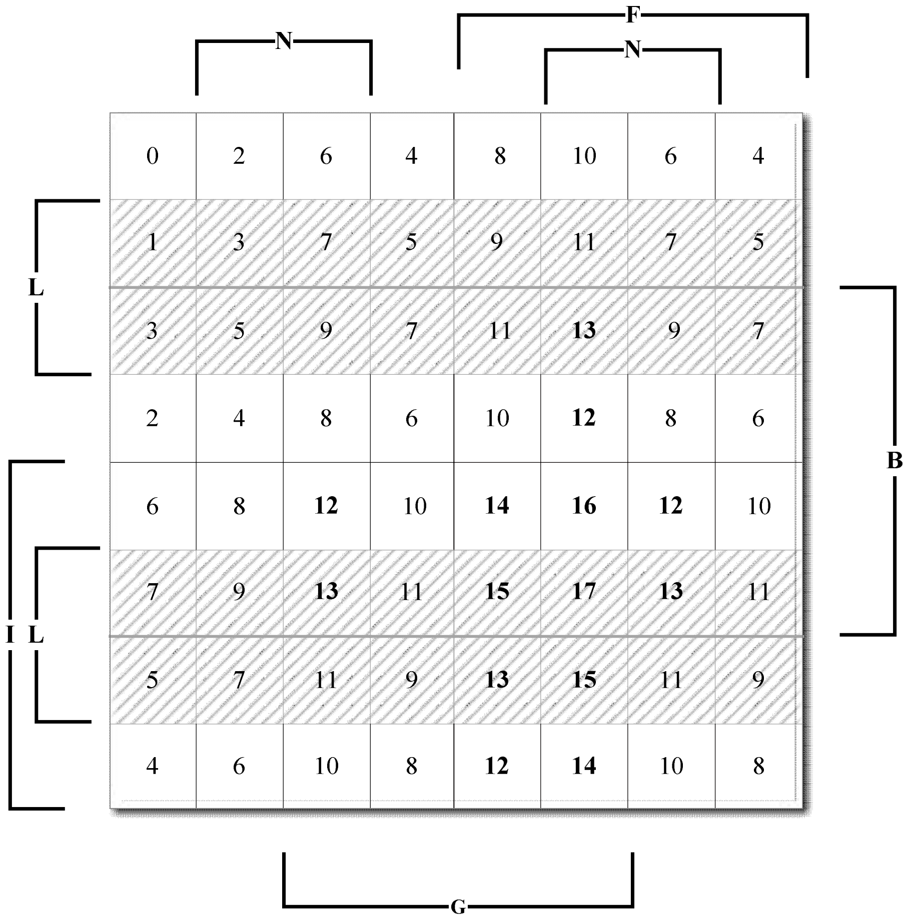
Table 1. Comparison between a yes-no voting system and a Boolean function describing its success

Property	**A yes-no voting system**	**A success Boolean function $g(\mathbf{X})$**
Causality	The *empty* coalition (the one to which none of the voting members belongs) is losing.	$g(\mathbf{0}) = 0$
	The *grand* coalition (the one to which all the voting members belong) is winning.	$g(\mathbf{1}) = 1$
Monotonicity	A winning coalition remains winning if a new voting member joins it and a losing coalition remains losing if one of its members defects.	$g(\mathbf{X}) \geq g(Y)$ for $X \geq (Y)$ or $g(X\|1_i) \geq g(X\|0_i)$
Relevancy	No voting member is dummy, *i.e.*, for every voting member, there exists at least one winning coalition to which this member belongs *critically*, i.e., such that the coalition ceases to be winning upon *defection* of this member.	$\frac{\partial g(X)}{\partial X_i} = g(X\|0_i) \oplus g(X\|1_i)$ is not identically 0, *i.e.*, there exists an instance of **X** such that $g(X\|1_i) \oplus g(X\|0_i) \neq 0$



$$G_1 = 4F + 4G + 4I + 2B + 2N + L.$$

Fig. 3. A replica of Fig. 2 but with the asserted region of B shaded. Arrows indicate when defection of B changes a WPC into a LPC. The number of these arrows is TBI(B) = 6

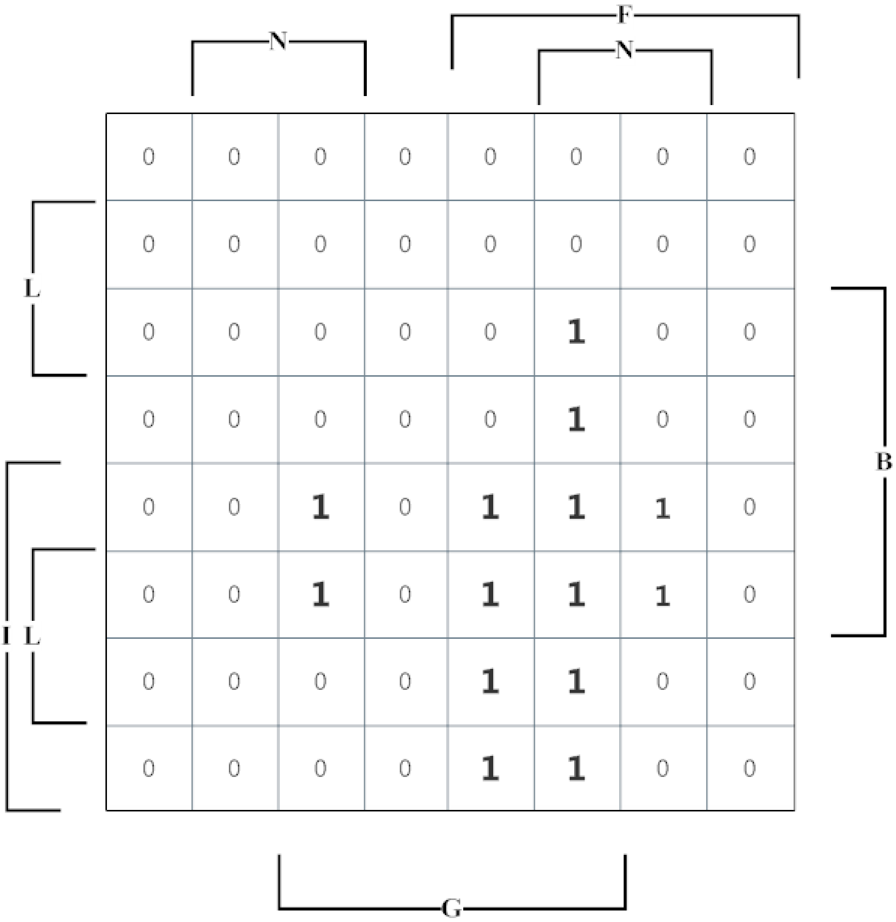


$$G_1 = 4F + 4G + 4I + 2B + 2N + L.$$

Fig. 4. A redrawing of Fig. 2 with the asserted region of L shaded. All mirror images across the L-boundary are either WPC-WPC or LPC-LPC. There is no transitions across the L-boundary from a WPC to a LPC, and hence TBI(L) = 0

Table 2. Various values of the Banzhaf indices for the old EEC

	F	G	I	B	N	L
Weights	4	4	4	2	2	1
Raw (Total) Banzahaf indices	10	10	10	6	6	0
Banzhaf indices normalized *w.r.t.* total combinations	$\frac{10}{64}$	$\frac{10}{64}$	$\frac{10}{64}$	$\frac{6}{64}$	$\frac{6}{64}$	$\frac{0}{64}$
Banzhaf indices normalized to add to 1	$\frac{10}{42}$	$\frac{10}{42}$	$\frac{10}{42}$	$\frac{6}{42}$	$\frac{6}{42}$	$\frac{0}{42}$



$g_1(F, G, I, B, N, L)$.

Fig. 5. A threshold switching function representing the EEC voting system. It is equal to 1 iff the inequality $\{G_1 \geq 12\}$ is satisfied, and equal to 0 otherwise. From now on, all 0-entered cells will be left blank

5 Vekm Utilization in the Computation of Banzhaf Indices

We now demonstrate the utility of the Variable-Entered Karnaugh Map (VEKM) in computing the Banzhaf indices of the $9-$member Extended European Economic Community (EEEC). Fig. 9 represents condition (5) as a Karnaugh map of 5 map variables depicting indicators for the less-weighted countries B, N, D, E, and L. Entries of the map are in terms of a weighted sum of the indicators of the more (and equally) weighted countries F, G, I, and R. Fig. 10 is a VEKM representation of the success $g_2(X)$ of the EEEC system. Entries of the VEKM in Fig. 10 are in terms of certain symmetric switching function [16,17]. The functions are of the form $Sy(C; Y)$, where $\mathbf{Y} = \{F, G, I, R\}$ and $C \subseteq \{0, 1, 2, 3, 4, 5, 6\}$ is the characteristic set of the function. In fact, C in Fig. 10 takes only 3 values, namely φ (corresponding to $Sy(\varphi; Y = 0)$, {4} or {3, 4}. The function $g_2(X)$ can now be read via standard VEKM techniques [18,19,20]. In fact, the entered function $Sy(\{3, 4\}; Y)$, is covered by the loops in Fig. 10, while the function $Sy(\{4\}; Y)$ appears everywhere in the map except in the all-0 cell, and is covered simply by a disjunction of all map variables. The expression for $g_2(X)$ is

$$g_2(X) = (BNL \vee BNE \vee BND \vee NED \vee BED)Sy(\{3,4\}; F, G, I, R) \vee (B \vee N \vee D \vee E \vee L)S_y(\{4\}; F, G, I, R)$$

$$= (BNL \vee BNE \vee BND \vee NED \vee BED)(FGI \vee FGR \vee FIR \vee GIR)$$

$$\vee (B \vee N \vee D \vee E \vee L)FGIR. \quad (10)$$

The above expression is the minimal sum (also the complete sum) of $g_2(X)$. It is a disjunction of 25 prime implicants each representing a minimal winning coalition that has a weight of 41 or more. Due to symmetry, we note that

$$TBI(F) = TBI(G) = TBI(I) = TBI(R), \quad (11a)$$

$$TBI(B) = TBI(N), \quad (11b)$$

$$TBI(D) = TBI(E), \quad (11c)$$

and hence it will suffice to calculate TBI for France (F), Belgium (B), Denmark (D), and Luxemburg (L). Figs. 11-14 are VEKM representations for $\frac{\partial g_2}{\partial F}$, $\frac{\partial g_2}{\partial B}$, $\frac{\partial g_2}{\partial D}$, and $\frac{\partial g_2}{\partial L}$, respectively. In Fig. 11, we made use of the relations

$$\frac{\partial}{\partial F}(Sy\{4\}; F, G, I, R)) = \frac{\partial}{\partial F}(FGIR) = GIR, \quad (12)$$

$$\frac{\partial}{\partial F}(Sy\{3,4\}; F, G, I, R)) = \frac{\partial}{\partial F}(FGI \vee FGR \vee FIR \vee GIR)$$
$$= (GI \vee GR \vee IR) \oplus (GIR)$$
$$= GI\overline{R} \vee G\overline{I}R \vee \overline{G}IR, \quad (13)$$

While in Figs. 12-14, we used the relation

$$Sy(\{4\}; F, G, I, R) \oplus Sy(\{3, 4\}; F, G, I, R) = Sy(\{3\}; F, G, I, R)$$
$$= \overline{F}GIR \vee F\overline{G}IR \vee FG\overline{I}R \vee FGI\overline{R}. \quad (14)$$

Thanks to the fact that every entry in the maps of Figs. 11-14 is in the form of a minterm expansion, we can immediately write the required indices as

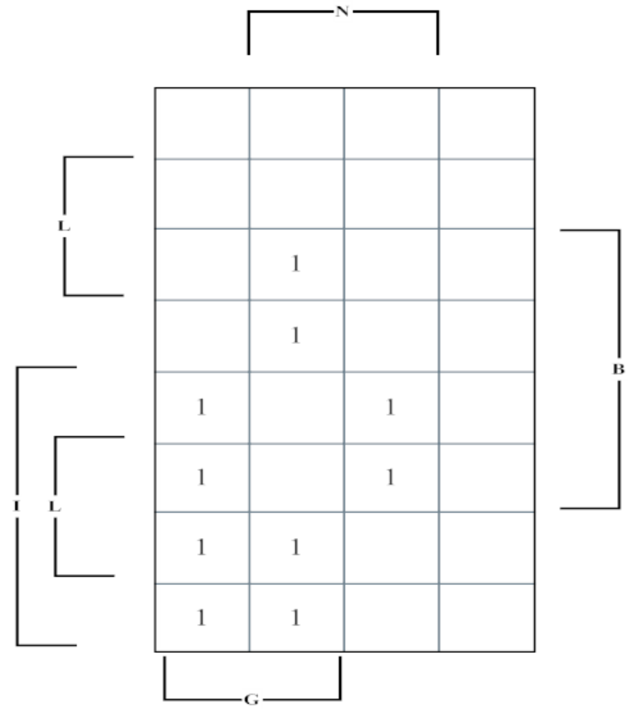
$$TBI(F) = weight\left(\frac{\partial g_2}{\partial F}\right) = 20 \times 1 + 11 \times 3 = 20 + 33 = 53, \quad (15a)$$

$$TBI(B) = weight\left(\frac{\partial g_2}{\partial B}\right) = 1 \times 1 + 7 \times 4 = 1 + 28 = 29, \quad (15b)$$

$$TBI(D) = weight\left(\frac{\partial g_2}{\partial D}\right) = 1 \times 1 + 5 \times 4 = 1 + 20 = 21, \quad (15c)$$

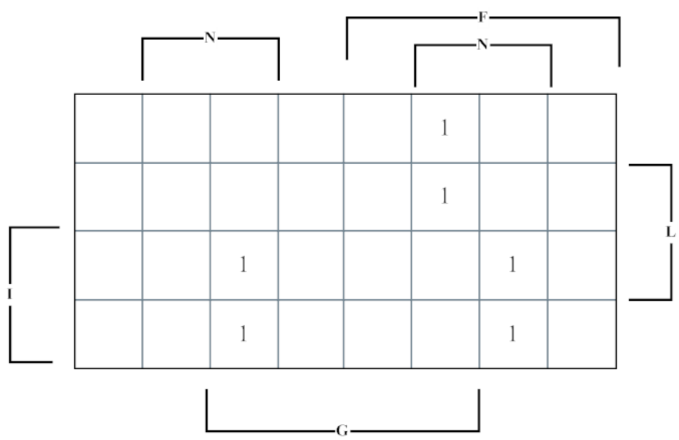
$$TBI(L) = weight\left(\frac{\partial g_2}{\partial L}\right) = 1 \times 1 + 1 \times 4 = 1 + 4 = 5, \quad (15d)$$

A notable observation is that L is no longer a dummy voter as it was in the old EEC. The indices might be normalized by dividing each TBI by the sum of the *TBIs*. Each of France, Germany, Italy and Britain has a normalized Banzhaf voting power of (53/317) = 0.167. The corresponding values are (29/317)= 0.091 for Belgium and the Netherlands, (21/317)= 0.066 for Denmark and Ireland and (5/317) = 0.016 for Luxembourg.



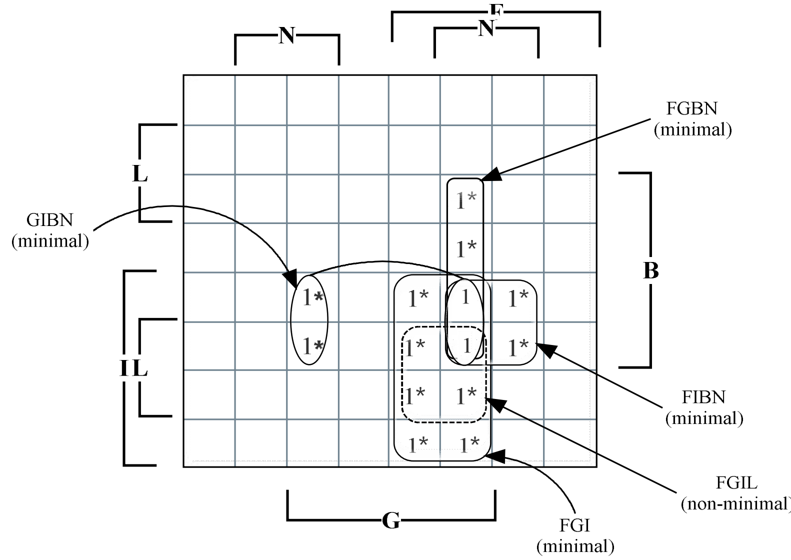
$$\frac{\partial g_1}{\partial F}$$

Fig. 6. The Boolean derivative ($\frac{\partial g_1}{\partial F}$) representing configurations where France (F) is critical within the EEC



$$\frac{\partial g_1}{\partial B}$$

Fig. 7. The Boolean derivative ($\frac{\partial g_1}{\partial B}$) representing configurations where Belgium (B) is critical within the EEC



$g_1(F, G, I, B, N, L)$.

Fig. 8. The minimal winning coalitions are the prime implicants of g_1, namely coalitions FGI, FGBN, FIBN, and GIBN of weight 12 each. Other (non-minimal) winning coalitions are among non-prime implicants of g_1, namely, those of weight 13 (FGIL, FGBNL, FIBNL, and GIBNL), those of weight 14 (FGIB, FGIN), those of weight 15 (FGIBL, FGINL), that of weight 16 (FGIBN) and the grand coalition of weight 17 (FGIBNL). Loops for all minimal winning coalitions are shown. A sample loop for a non-minimal winning coalition (FGIL) is also shown. Each loop of a winning coalition (whether minimal or not) has a product of non-complemented literals only and hence must cover the all-1 shaded cell that represents the grand coalition

$A \geq 41$	$A+5 \geq 41$	$A+10 \geq 41$	$A+5 \geq 41$
$A+2 \geq 41$	$A+7 \geq 41$	$A+12 \geq 41$	$A+7 \geq 41$
$A+5 \geq 41$	$A+10 \geq 41$	$A+15 \geq 41$	$A+10 \geq 41$
$A+3 \geq 41$	$A+8 \geq 41$	$A+13 \geq 41$	$A+8 \geq 41$
$A+6 \geq 41$	$A+11 \geq 41$	$A+16 \geq 41$	$A+11 \geq 41$
$A+8 \geq 41$	$A+13 \geq 41$	$A+18 \geq 41$	$A+13 \geq 41$
$A+5 \geq 41$	$A+10 \geq 41$	$A+14 \geq 41$	$A+10 \geq 41$
$A+3 \geq 41$	$A+8 \geq 41$	$A+13 \geq 41$	$A+8 \geq 41$

Fig. 9. Conditions for the Expanded European Economic Community (EEEC), where A = 10(F+G+I+R)

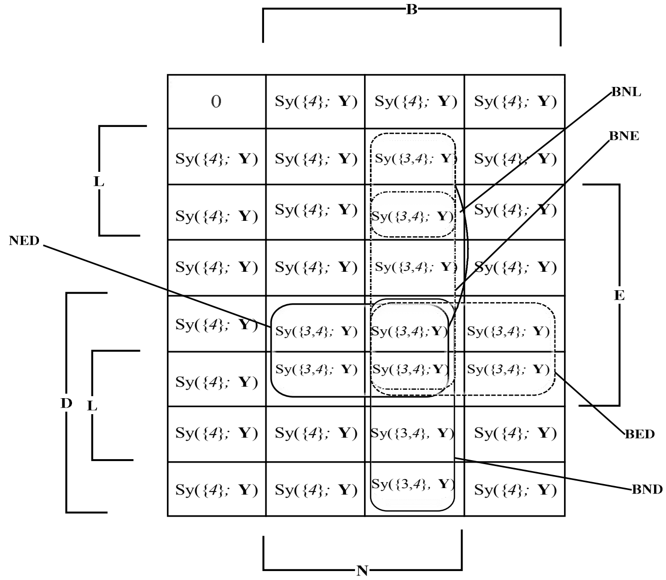
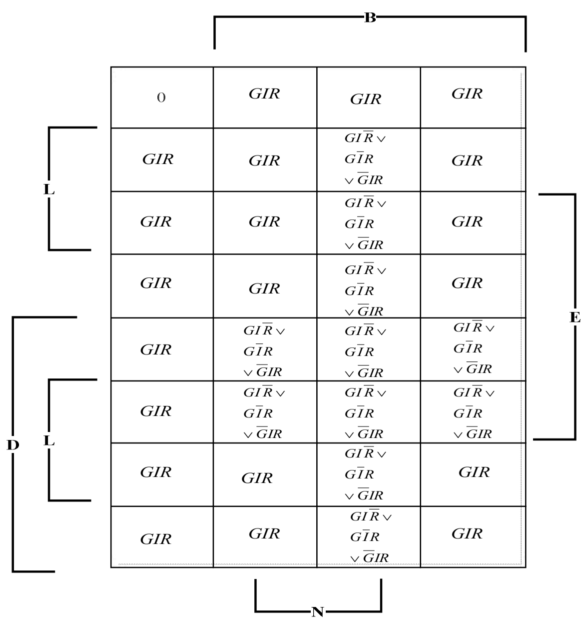


Fig. 10. Threshold function representing the upholding of a decision on the Expanded European Economic Community (EEEC)



$$\frac{\partial g_2}{\partial F}$$

Fig. 11. The Boolean derivative $\frac{\partial g_2}{\partial F}$ representing when France is critical within the EEEC

$FGIR$	0
0	$\overline{F}GIR \vee F\overline{G}IR \vee FG\overline{I}R \vee FGI\overline{R}$
0	$\overline{F}GIR \vee F\overline{G}IR \vee FG\overline{I}R \vee FGI\overline{R}$
0	$\overline{F}GIR \vee F\overline{G}IR \vee FG\overline{I}R \vee FGI\overline{R}$
$\overline{F}GIR \vee F\overline{G}IR \vee FG\overline{I}R \vee FGI\overline{R}$	0
$\overline{F}GIR \vee F\overline{G}IR \vee FG\overline{I}R \vee FGI\overline{R}$	0
0	$\overline{F}GIR \vee F\overline{G}IR \vee FG\overline{I}R \vee FGI\overline{R}$
0	$\overline{F}GIR \vee F\overline{G}IR \vee FG\overline{I}R \vee FGI\overline{R}$

L L D E N

Fig. 12. The Boolean derivative $\frac{\partial g_2}{\partial B}$ representing when Belgium (B) is critical within the EEEC

B

$FGIR$	0	$\overline{F}GIR \vee F\overline{G}IR \vee FG\overline{I}R \vee FGI\overline{R}$	0
0	0	0	0
0	$FGIR \vee FGIR \vee FG\overline{I}R \vee FGI\overline{R}$	0	$\overline{F}GIR \vee F\overline{G}IR \vee FG\overline{I}R \vee FGI\overline{R}$
0	$\overline{F}GIR \vee F\overline{G}IR \vee FG\overline{I}R \vee FGI\overline{R}$	0	$\overline{F}GIR \vee F\overline{G}IR \vee FG\overline{I}R \vee FGI\overline{R}$

L E N

$$\frac{\partial f}{\partial D}$$

Fig. 13. The Boolean derivative $\frac{\partial g_2}{\partial D}$ representing when Denmark (D) is critical within the EEEC

B (columns 3–4); D (rows 2–4, left); E (rows 2–4, right); N (columns 2–3)

$FGIR$	0	$\bar{F}GIR \vee F\bar{G}IR \vee FG\bar{I}R \vee FGI\bar{R}$	0
0	0	0	0
0	0	0	0
0	0	0	0

$$\frac{\partial g_2}{\partial L}$$

Fig. 14. The Boolean derivative $\frac{\partial g_2}{\partial L}$ representing when Luxemburg (L) is critical within the EEEC. Note that Luxemburg ceased to be dummy or powerless upon expansion of the EEC

6 Conclusion

This paper is mainly intended as a demonstration of utility the of the VEKM in the computation of Banzhaf indices. The demonstration is achieved via a 9-variable example using a VEKM of 5 map variables and 4 entered variables. However, the VEKM might be conveniently used up to 12 variables (6 map variables plus 6 entered ones). As an offshoot, the paper also has a strong pedagogical and expository component as it translates many concepts and results of weighted voting systems to the Boolean domain.

In particular, system states and configurations are represented as individual cells called primitive coalitions. The system minimal winning coalitions are identified with the prime implicants of the system success function. Non-minimal winning coalitions are also related to minimal ones via partial-order structures and also as particular subordinate loops that cover the all-1 cell in the Karnaugh map.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Taylor AD, Pacelli AM. Mathematics and politics: Strategy, voting, power and proof. Second Edition, Springer Science+Business Media, New York, NY, USA; 2008.

[2] Banzhaf JF. Weighted voting doesn't work: A mathematical analysis, Rutgers Law Review. 1964;19: 317-343.

[3] Kirsch W, Jessica L. Power indices and minimal winning coalitions. Social Choice and Welfare. 2010;34(1):33-46.

[4] Felsenthal DS, Machover M. Alternative forms of the Shapley value and the Shapley-Shubik index. Public Choice. 1996;87(3-4):315-318.

[5] Rushdi AMA, Ba-Rukab OM. Map calculation of the Shapley-Shubik voting powers: An example of the European Economic Community. International Journal of Mathematical, Engineering and Management Sciences (IJMEMS). 2017;2(1):17-29.

[6] Lee SC. Modern switching theory and digital design. Prentice-Hall, Englewood Cliffs, New Jersey, NJ, USA; 1978.

[7] Rushdi AMA, Alturki AM. Reliability of coherent threshold systems. Journal of Applied Science. 2015;15(3):431-443.

[8] Crama Y, Hammer PL. Boolean functions: Theory, algorithms, and applications. Cambridge University Press, Cambridge, United Kingdom; 2011.

[9] Muroga S. Logic design and switching theory. John Wiley & Sons, New York, NY, USA; 1979.

[10] Rushdi AM. Threshold systems and their reliability, Microelectronics and Reliability. 1990;30(2): 299-312.

[11] Alturki AM, Rushdi AM. Weighted voting systems: A threshold-Boolean perspective. Journal of Engineering Research. 2016;4(1):125-143.

[12] Felsenthal DS, Machover M. A priori voting power: What is it all about?. Political Studies Review. 2004;2(1):1-23.

[13] Strafiin Jr, PD. The Shapley—Shubik and Banzhaf power indices as probabilities. In Roth A. E. (Editor), The Shapley Value: Essays in Honor of Lloyd S. Shapley, Press Syndicate of the University of Cambridge, Cambridge, England. 1988;71-81.

[14] Rushdi AM. Map differentiation of switching functions. Microelectronics and Reliability. 1986;25(5): 891-908.

[15] Yamamoto Y, Banzhaf index and Boolean difference. Proceedings of the 42th IEEE International Symposium on Multiple-Valued Logic (ISMVL). 2012;191-196.

[16] Rushdi AM. Utilization of symmetric switching functions in the computation of k-out-of-n system reliability. Microelectronic and Reliability. 1986;26(5):973-987.

[17] Rushdi AM. Reliability of k-out-of-n systems. In K. B. Misra (Editor), New Trends in System Reliability Evaluation, Elsevier, Amsterdam. 1993;16:185-227.

[18] Rushdi AM. Improved Variable-Entered Karnaugh Map procedures. Computers and Electrical Engineering. 1987;13(1):41-52.

[19] Rushdi AM, Al-Yahya HA. A Boolean minimization procedure using the variable-entered Karnaugh map and the generalized consensus concept. International Journal of Electronics. 2000;87(7):769-794.

[20] Rushdi AM, Al-Yahya HA. Further improved variable-entered Karnaugh map procedures for obtaining the irredundant forms of an incompletely-specified switching function. Journal of King Abdulaziz University: Engineering Sciences. 2001;13(1):111-152.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/17676