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Existence and Uniqueness of Positive Periodic Solution of an Extended Rosenzweig-MacArthur Model via Brouwer's Topological Degree

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Authors' contributions

This work was carried out in collaboration between all authors. Author EEJ formulated a dimensionless dynamical system and generated its numerical simulations, author ETA used Brouwer's and coincidence degree algorithm to established existence of solutions. Authors OA and CEM proved the uniqueness of the periodic solution. The final manuscript was completed by author ETA. All authors read and approved the final manuscript.

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Abstract

The necessary conditions for existence of periodic solutions of an Extended Rosenzweig-MacArthur model are obtained using Brouwer's degree. The forward invariant set is formulated to ensure the boundedness of the solutions, using Brouwers fixed point properties, and Zorns lemma. Also, sufficient conditions for the existence of a unique positive periodic solution have been established using Barbalats lemma and Lyapunovs functional. Numerical responses show that, the phase-flows of the non-autonomous system exhibit an asymptotically stable periodic solution which is globally attractive and trapped in the absorbing region.

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1 Introduction

Mathematical modelling of ecological system has explored robust modifications in terms of the nature of their interactions (i.e., competitive, prey-predator systems, spatio-temporal dynamics, coope- rative systems, patch-diffusion, delay systems and so on), functional responses (i.e., Holling types, Leslie-Gower, Beddington-DeAngelis, and so on) and ecologically perturbative parameters. In prey-predator systems, it is pertinent to assume that all biological and environmental perturbative parameters and state variables are subject to natural fluctuations in time. Thus, the assumption of periodically varying perturbative parameters is a way of making the dynamical system more realistic as compared to constant perturbative parameters. Obviously, periodic variations in the environment and ecologically perturbative parameters are characterized by seasonal effect of weather, food supplies, predation effects, mating durations, time delay due to gestation, and so on.

The qualitative dynamical behaviors of these mathematical models are widely studied in populations of multiple interacting species in the ecosystem. [1] investigated the existence and global attractivity of positive periodic solutions for a Holling II two-prey and one-predator system. Periodic solutions for a three-species Lotka-Volterra food chain model with time delay were studied in [2]. They derived the sufficient conditions for the existence of positive periodic solutions of the system. In the same [3], obtained the necessary and sufficient conditions for existence of periodic solutions of predator-prey dynamical system with Beddington-DeAngelis-type functional response. Existence of periodic solutions for a two-species non-autonomous competitive Lotka-Volterra patch system with time delay was established in [4].

Exploration of these robust dynamical systems requires using topological degree theory, see [5] [6] [7]. In this theory, to prove the existence of solution for a natural abstract formulated IVP, say

$$\begin{cases} \ddot{X} = F(t, \ddot{X}(t), \dot{X}, X(t); t \in [0, \omega] \\ F \subset C^1 : [0, \omega] \times \mathbb{R}^3 \to \mathbb{R}^3 \\ X(0) = X(\omega), \dot{X}(0) = \dot{X}(\omega), \ddot{X}(0) = \ddot{X}(\omega), X(t) = X(t + \omega) \end{cases} \tag{1.1}$$

usually reduces to solving the abstract operator equation, $L(X) = N(X)$ which has some topological degree properties, see [8]. Moreover, results of theorems, and propositions well established via Topological Degree Theory can be numerically simulated using sophisticated dynamical tools (e.g MAPLE)[9] [10].

2 Model Formulation and Its Invariance Region

The Extended Rosenzweig-MacArthur Model formulated and studied in [11] is given as:

$$\begin{cases} \dfrac{dx_1}{dt} &= rx_1 - \dfrac{rx_1^2}{K} - a_2 \dfrac{x_1}{b_1 + x_1} x_2 - a_3 \dfrac{x_1}{b_1 + x_1} x_3 \\ \dfrac{dx_2}{dt} &= c_2 a_2 \dfrac{x_2}{b_1 + x_1} x_3 - d_2 x_2 - a_3 \dfrac{x_2}{b_2 + x_2} x_3 \\ \dfrac{dx_3}{dt} &= c_3 a_3 \dfrac{x_2}{b_2 + x_2} x_3 - d_3 x_3 + c_3 a_3 \dfrac{x_1}{b_1 + x_1} x_3 \end{cases} \tag{2.1}$$

where $x_1(t), x_2(t)$, and $x_3(t)$ are the population densities of the interacting species and $r, K, a_2, a_3, b_1, b_2, c_2, c_3, d_2$ and d_3 are positive ecological parameters. In [12] a topologically equivalent dynamical model of system (2.1) was obtained via non-dimensionalization of the state variables as follows:

$$\begin{cases} \dfrac{dx}{d\tau} &= \alpha u - \dfrac{\alpha u^2}{\kappa} - \eta\dfrac{u}{1+u}v - \dfrac{u}{1+u}w \\ \dfrac{dy}{d\tau} &= \varepsilon\dfrac{u}{1+u}v - \xi v - \sigma\dfrac{v}{1+v}w \\ \dfrac{dz}{d\tau} &= \beta\dfrac{v}{1+v}w - \mu w + \beta\dfrac{u}{1+u}w \end{cases} \tag{2.2}$$

where $x(\tau) = \dfrac{x_1(t)}{b_1}, y(\tau) = \dfrac{x_2(t)}{b_2}, z(\tau) = \dfrac{x_3(t)}{b_1}, \alpha = \dfrac{r}{a_3}, \kappa = \dfrac{K}{b_1}, \eta = \dfrac{a_2b_2}{a_3b_1}, \varepsilon = \dfrac{c_2a_2}{a_3}, \xi = \dfrac{d_2}{a_3}, \sigma = \dfrac{b_1}{b_2}, \mu = \dfrac{d_3}{a_3}, \tau = a_3t, c_3 = \beta$. Suppose the ecological parameters are periodic functions, so system (2.2) can be modified to a non-autonomous system as follows:

$$\begin{cases} \dfrac{du}{d\tau} &= \alpha(\tau) - \dfrac{\alpha(\tau)\exp u(\tau)}{\kappa(\tau)} - \eta(\tau)\dfrac{\exp v(\tau)}{1+\exp u(\tau)} - \dfrac{\exp w(\tau)}{1+\exp u(\tau)} \\ \dfrac{dv}{d\tau} &= \varepsilon(\tau)\dfrac{\exp u(\tau)}{1+\exp u(\tau)} - \xi(\tau) - \sigma(\tau)\dfrac{\exp w(\tau)}{1+\exp v(\tau)} \\ \dfrac{dw}{d\tau} &= \beta(\tau)\dfrac{\exp v(\tau)}{1+\exp v(\tau)} - \mu(\tau) + \beta(\tau)\dfrac{\exp u(\tau)}{1+\exp u(\tau)} \end{cases} \tag{2.3}$$

where $u(\tau) = In \mid x(\tau) \mid, v(\tau) = In \mid y(\tau) \mid, w(\tau) = In \mid z(\tau) \mid, \alpha(\tau) = \alpha(\tau+\omega), \eta(\tau) = \eta(\tau+\omega), \varepsilon(\tau) = \varepsilon(\tau+\omega), \xi(\tau) = \xi(\tau+\omega), \sigma(\tau) = \sigma(\tau+\omega), \beta(\tau) = \beta(\tau+\omega), \mu(\tau) = \mu(\tau+\omega)$, and subject to initial conditions, $u(0) = u_0 > 0, v(0) = v_0 > 0, w(0) = w_0 > 0$.

Using the fundamental theorem of calculus, it is easy to see that $\mathbb{R}^3_+$ is the invariance region of solutions of system (2.3) satisfying;

$$\begin{cases} u(\tau) = u_o \exp\int_0^\omega \{\alpha(\tau) - \dfrac{\alpha(\tau)\exp u(\tau)}{\kappa(\tau)} - \eta(\tau)\dfrac{\exp v(\tau)}{1+\exp u(\tau)} - \dfrac{\exp w(\tau)}{1+\exp u(\tau)}\}ds \\ v(\tau) = v_0 \exp\int_0^\omega \{\varepsilon(\tau)\dfrac{\exp u(\tau)}{1+\exp u(\tau)} - \xi(\tau) - \sigma(\tau)\dfrac{\exp w(\tau)}{1+\exp v(\tau)}\}ds \\ w(\tau) = w_0 \exp\int_0^\omega \{\beta(\tau)\dfrac{\exp v(\tau)}{1+\exp v(\tau)} - \mu(\tau) + \beta(\tau)\dfrac{\exp u(\tau)}{1+\exp u(\tau)}\}ds \end{cases} \tag{2.4}$$

Thus, the state variables are invariants in the positive octant cone, $\mathbb{R}^3_+ = (((u\tau), v(\tau), w(\tau))^T \in \mathbb{R}^3 : u(\tau) > 0, v(\tau) > 0, w(\tau) > 0)$.

3 Some Results on Brouwer's Topological Degree Theory

3.1 Lemma 1 [13]

Assume $f : \mathbb{T} \subset \mathbb{R} \to \mathbb{R}$ is $\omega-$periodic function, let $\tau_1, \tau_1 \in [0, \omega]$ then, $\bar{f} = \frac{1}{\omega}\int_0^\omega \mid f(\tau) \mid d(\tau), f^l = minf(\tau), f^m = maxf(\tau)$

$\forall \tau \in [0,\omega]$ and

$$\begin{cases} f(\tau) \leq f(\tau_1) + \int_0^{\omega} \mid \dot{f}(s) \mid ds \\ f(\tau) \geq f(\tau_2) - \int_0^{\omega} \mid \dot{f}(s) \mid ds \end{cases} \tag{3.1}$$

3.2 Lemma 2 [8]

Let X and Y be two Banach spaces and let $L : DomL \subset X \to Y$ be a linear operator. Let $N : X \to Y$ be a continuous mapping. A mapping $F : DomL \subset X \to Y$ is said to be a Fredholm mapping of index zero, if $dimKerL = codimImL < \infty$ and ImL is closed in Y. If L is a Fredholm mapping, its index is an integer $IndL = dimL - codimImL$. Suppose L is a Fredholm mapping of index zero, there exist continuous projections, $P : X \to X$and $Q : Y \to Y$ such that $ImP = KerL, ImL = KerQ = Im(I-Q)$, and the restriction L_P of L to $DomL \cap KerP : (I-Q)X \to ImL$ is invertible. Denote the generalized inverse of L_P by K_P such that $LK_P = I$, and $K_PL = I - P$. Let Ω be a non-empty, open bounded subset of X, then the mapping N is said to be L-compact on $\bar{\Omega}$ if the mapping $QN : \Omega \to Y$ is continuous, $QN(\bar{\Omega})$ is bounded, and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact (i.e., it is continuous, and $K_P(I-Q)N(\bar{\Omega})$ relatively compact). Since ImQ is isomorphic to $KerL$, then there exists an isomorphism $J : ImQ \to KerL$.

3.3 Lemma 3 [7]

Let $\Omega \in \mathbb{R}^n$ be an open bounded set and $L : \bar{\Omega} \to R^n$ be a continuous mapping. If $p \notin L(\partial\Omega)$, then the Brouwer degree of L at p relative to Ω is an integer number, denoted by: $deg(L, \Omega, p) = sign \mid J_p(p) \mid$, where $J_P(p)$ is the Jacobian matrix of the operator L at p, satisfying the following properties:

- i $deg(I, \Omega, p) = 1$, iff $p \in \Omega$, where I denotes the identity mapping.
- ii if $deg(L, \Omega, p) \neq 0$ then $Lx = p$ has a solution in Ω
- iii if $H(t, x) : [0,1] \times \bar{\Omega} \to R^n$ is a continuous homotopic mapping defined as $H(t, x) = t\phi(x) + (1-t)\psi(x)$ for $\phi, \psi \in C^1(\Omega)$, and $\forall p \in \mathbb{R}^n \setminus H(t, \partial\Omega)$, then $deg(\phi, \Omega, p) = deg(\psi, \Omega, p)$ and $deg(H(t, x), \Omega, p) = deg(H(0, x), \Omega, p)$ independent of $t \in [0, 1]$.

3.4 Lemma 4 [5]

Let Ω be an open bounded set. Let L be a Fredholm mapping of index zero, and N be $L-compact$ on $\bar{\Omega}$. Assume

- i for each $t \in (0, 1)$ every solution x of $Lx = tNx$, is such that $x \notin DomL \cap \partial\Omega$.
- ii $QNx \neq 0, \forall x \in DomL \cap \partial\Omega$, and
- iii $deg(JQN : KerL \cap \Omega, 0) \neq 0$ where $J : ImQ \to KerL$ is an Isomorphism and deg denotes the Brouwer topological degree.

Then the operator equation, $Lx = Nx$ has at least one solution in $DomL \cap \partial\Omega$.

4 Existence of Positive Periodic Solutions

4.1 Proposition 1

Assuming that the perturbation parameters of system (2.3) are periodic functions, then system (2.3) has at least one positive periodic solution.

Proof: Suppose $X = Y = (u(\tau), v(\tau), w(\tau))^T \in C_c^1(\mathbb{R}, \mathbb{R}^3) : u(\tau) = u(\tau + \omega), v(\tau) = v(\tau + \omega), w(\tau) = w(\tau + \omega)$ is the phase flows system (2.3), then equipped the spaces, X and Y with the usual Euclidean norm, say $\| u(\tau), v(\tau), w(\tau) \| = max \mid u(\tau) \mid + max \mid v(\tau) \mid + max \mid w(\tau) \mid \forall \tau \in [0, \omega]$. Denote $L : DomL \subset X \to Y$ and $N : X \to Y$ as operator equations,

$$L(u(\tau), v(\tau), w(\tau))^T = (\dot{u}(\tau), \dot{v}(\tau), \dot{w}(\tau))$$

$$N(u(\tau), v(\tau), w(\tau))^T = \begin{cases} \alpha(\tau) - \dfrac{\alpha(\tau) \exp u(\tau)}{\kappa(\tau)} - \eta(\tau) \dfrac{\exp v(\tau)}{1 + \exp u(\tau)} - \dfrac{\exp w(\tau)}{1 + \exp u(\tau)} \\ \varepsilon(\tau) \dfrac{\exp u(\tau)}{1 + \exp u(\tau)} - \xi(\tau) - \sigma(\tau) \dfrac{\exp w(\tau)}{1 + \exp v(\tau)} \\ \beta(\tau) \dfrac{\exp v(\tau)}{1 + \exp v(\tau)} - \mu(\tau) + \beta(\tau) \dfrac{\exp u(\tau)}{1 + \exp u(\tau)} \end{cases} \tag{4.1}$$

Define two continuous projectors $P : X \to X$ and $Q : Y \to Y$ as

$$P(u(\tau), v(\tau), w(\tau))^T = Q(u(\tau), v(\tau), w(\tau))^T = \begin{cases} \frac{1}{\omega} \int_0^\omega u(\tau) d\tau \\ \frac{1}{\omega} \int_0^\omega v(\tau) d\tau, \quad (u(\tau), v(\tau), w(\tau))^T \in X = Y \\ \frac{1}{\omega} \int_0^\omega w(\tau) d\tau \end{cases}$$

It is clear that $KerL = (\mathbf{x} \in X : \mathbf{x} = \mathbf{h}, \mathbf{h} \in \mathbb{R}^3)$, and $ImL = (\mathbf{y} \in Y : \int_0^\omega \mathbf{y}(\tau) d\tau = 0)$ is closed in Y. Observe that $dimKerL = codimImL = 3$, $ImP = KerL$, $KerQ = ImLQ = Im(I - Q)$ Therefore, L is a Fredholm mapping of index zero.
Furthermore, the generalized inverse K_P of L_P has the form $K_P : ImL \to DomL \cap KerP$,

$$K_p y = \int_0^\tau y(s) ds - \tfrac{1}{\omega} \int_0^\omega \int_0^\tau y(s) ds d\tau$$

Then, $QN : X \to Y$ yields

$$QN\mathbf{x} = \begin{cases} \frac{1}{\omega} \int_0^\omega (\alpha(\tau) - \dfrac{\alpha(\tau) \exp u(\tau)}{\kappa(\tau)} - \eta(\tau) \dfrac{\exp v(\tau)}{1 + \exp u(\tau)} - \dfrac{\exp w(\tau)}{1 + \exp u(\tau)}) d\tau \\ \frac{1}{\omega} \int_0^\omega (\varepsilon(\tau) \dfrac{\exp u(\tau)}{1 + \exp u(\tau)} - \xi(\tau) - \sigma(\tau) \dfrac{\exp w(\tau)}{1 + \exp v(\tau)}) d\tau \\ \frac{1}{\omega} \int_0^\omega (\beta(\tau) \dfrac{\exp v(\tau)}{1 + \exp v(\tau)} - \mu(\tau) + \beta(\tau) \dfrac{\exp u(\tau)}{1 + \exp u(\tau)}) d\tau \end{cases}$$

and $K_p(I - Q)N : X \to X$ yields

$$K_p(I - Q)N\mathbf{x} = \int_0^\tau N\mathbf{x} ds - \frac{1}{\omega} \int_0^\omega \int_0^\tau N\mathbf{x} ds d\tau - \frac{1}{\omega} \int_0^\tau \int_0^\omega N\mathbf{x} ds ds + \frac{1}{\omega^2} \int_0^\omega \int_0^\tau \int_0^\omega N\mathbf{x} ds ds d\tau$$

Clearly, by Lebesgue convergence theorem, QN and $K_P(I - Q)N$ are continuous maps. Since the maps are well-defined on finite dimensional Banach spaces, by Arzela-Ascoli theorem, $K_P(I - Q)N(\bar{\Omega})$ is relatively compact. Additionally, $QN(\bar{\Omega})$ is bounded for any open bounded set $\Omega \subset X$, and N is $L - compact$.

We now seek a forward invariance set $K \subset X$ that is convex and compact such that the phase flows $\Phi(\tau) \subset K$ satisfy the operator equation $Lx = tNx, t \in (0, 1)$. Consider

$$\begin{cases} \dot{u}(\tau) = \alpha(\tau) - \dfrac{\alpha(\tau) \exp u(\tau)}{\kappa(\tau)} - \eta(\tau) \dfrac{\exp v(\tau)}{1 + \exp u(\tau)} - \dfrac{\exp w(\tau)}{1 + \exp u(\tau)} \\ \dot{v}(\tau) = \varepsilon(\tau) \dfrac{\exp u(\tau)}{1 + \exp u(\tau)} - \xi(\tau) - \sigma(\tau) \dfrac{\exp w(\tau)}{1 + \exp v(\tau)} \\ \dot{w}(\tau) = \beta(\tau) \dfrac{\exp v(\tau)}{1 + \exp v(\tau)} - \mu(\tau) + \beta(\tau) \dfrac{\exp u(\tau)}{1 + \exp u(\tau)} \end{cases} \tag{4.2}$$

integrating yields

$$\begin{cases} \omega\bar{\alpha} = \int_0^{\omega}(\alpha(\tau) - \dfrac{\alpha(\tau)\exp u(\tau)}{\kappa(\tau)} - \eta(\tau)\dfrac{\exp v(\tau)}{1+\exp u(\tau)} - \dfrac{\exp w(\tau)}{1+\exp u(\tau)})d\tau \\ \omega\bar{\xi} = \int_0^{\tau}(\varepsilon(\tau)\dfrac{\exp u(\tau)}{1+\exp u(\tau)} - \xi(\tau) - \sigma(\tau)\dfrac{\exp w(\tau)}{1+\exp v(\tau)})d\tau \\ \omega\bar{\mu} = \int_0^{\omega}(\beta(\tau)\dfrac{\exp v(\tau)}{1+\exp v(\tau)} - \mu(\tau) + \beta(\tau)\dfrac{\exp u(\tau)}{1+\exp u(\tau)})d\tau \end{cases} \tag{4.3}$$

and

$$\begin{cases} \int_0^{\omega} \mid \dot{u}(\tau) \mid d\tau \leq \int_0^{\omega}(\alpha(\tau)+ \mid \alpha(\tau) \mid)d\tau = \omega\overline{(\alpha + \mid \alpha \mid)} \\ \int_0^{\omega} \mid \dot{v}(\tau) \mid d\tau \leq \int_0^{\omega}\xi(\tau)+ \mid \xi(\tau) \mid d\tau) = \omega\overline{(\xi + \mid \xi \mid)} \\ \int_0^{\omega} \mid \dot{w}(\tau) \mid d\mu \leq \int_0^{\omega}\mu(\tau)+ \mid \mu(\tau) \mid d\tau) = \omega\overline{(\mu + \mid \mu \mid)} \end{cases} \tag{4.4}$$

Using Mean-Value Theorem for integral equations, we have that there exists $\delta_i \in [0,\omega]$ for $i = 1,2,3$ such that $u(\delta_1) \leq R_1, v(\delta_1) \leq R_2, w(\delta_1) \leq R_3$ where R_1, R_2, R_3 are sufficiently large. Using lemma 1, system (4.4) and proposition (1.6) from [14], the forward invariance region of system (2.3) is as follows:

$$\begin{cases} \mid u(\tau) \mid \leq \mid u(\delta_1) \mid + \int_0^{\omega} \mid \dot{u}(\tau) \mid d\tau < R_1 + \omega\overline{(\alpha + \mid \alpha \mid)} = M_1 \\ \mid v(\tau) \mid \leq \mid v(\delta_1) \mid + \int_0^{\omega} \mid \dot{v}(\tau) \mid d\tau < R_2 + \omega\overline{(\xi + \mid \xi \mid)} = M_2 \\ \mid w(\tau) \mid \leq \mid w(\delta_1) \mid + \int_0^{\omega} \mid \dot{w}(\tau) \mid d\tau < R_3 + \omega\overline{(\mu + \mid \mu \mid)} = M_3 \end{cases}$$

Observe that the set $K = [0, M_1]\times[0, M_2]\times[0, M_3]$ is forward invariance, compact and convex. Using Brouwer fixed point theorem, see [15], the phase flows $\Phi(\tau)$ of system (2.3) have at least a fixed point say, $(u^*, v^*, w^*) \in X$ such that $\Phi(\tau) \to (u^*, v^*, w^*)$ as $\tau \to \infty$. By Zorns lemma, and semi-group properties of phase flows $\Phi(\tau)$ of system (2.3), see [16], there exists a maximal element $\mathbb{M}$ satisfying $\|(u^*, v^*, w^*)^T\| = \mid u^* \mid + \mid v^* \mid + \mid w^* \mid < \mathbb{M}$, where $\mathbb{M} = M_1 + M_2 + M_3 + 1$ which is independent of the perturbation parameter $t \in (0,1)$. Taking $\Omega = (u(\tau), v(\tau), w(\tau))^T \in X : \|u, v, w\| < \mathbb{M}$; then it is easy to claim that Ω is an open bounded set in X, which verifies lemma 4 (i). When $u(\tau), v(\tau), w(\tau))^T \in \partial\Omega \cap KerL = \partial\Omega \cap R^3; (u(\tau), v(\tau), w(\tau))^T$ is a constant vector in R^3 with $\mid u \mid + \mid v \mid + \mid w \mid = \mathbb{M}$ and the operator equation $QNx \neq 0$ which verifies lemma 4(ii). We now verify lemma 4(iii) using lemma (3) as follows. Define a homotopic mapping, say $H(u, v, w; \lambda)$: $DomL \times [0,1] \to X$ by $H(u, v, w; \lambda) = \lambda\phi(u, v, w) + (1-\lambda)\psi(u, v, w)$ for $\lambda \in [0,1]$, where

$$\psi(u, v, w)^T = \begin{cases} \bar{\alpha} - \overline{\dfrac{1}{\kappa}}\exp u(\tau) \\ \bar{\varepsilon}\dfrac{\exp u(\tau)}{1+\exp u(\tau)} - \bar{\sigma}\dfrac{\exp w(\tau)}{1+\exp v(\tau)} \\ \dfrac{\bar{\beta}\exp v(\tau)}{1+\exp v(\tau)} - \bar{\mu} \end{cases} \tag{4.5}$$

Moreover, it can be easily shown that the approximated algebraic system (4.5) has a unique fixed point $(u^*, v^*, w^*) \in X \subset \mathbb{R}^3$ if $\bar{\beta} > \bar{\mu}$. Using homotopy invariance properties of Brouwer's degree, and taking $J = I : ImQ \to KerL$ then,

$deg(JQN(\Phi); KerL \cap \Omega, (0,0,0)^T) = deg(IQN(\Phi); KerL \cap \Omega, (0,0,0)^T = deg(\phi(u, v, w)^T; KerL \cap \Omega, (0,0,0)^T$

$$= sign\begin{vmatrix} -\overline{\dfrac{1}{\kappa}}\exp u(\tau) & 0 & 0 \\ \bar{\varepsilon}\dfrac{\exp(u(\tau)}{(1+\exp u(\tau))^2} & \bar{\sigma}\dfrac{(\exp w(\tau)}{(1+\exp v(\tau))^2} & -\bar{\sigma}\dfrac{\exp w(\tau)}{1+\exp v(\tau)} \\ 0 & \dfrac{\bar{\beta}\exp v(\tau)}{(1+\exp v(\tau))^2} & 0 \end{vmatrix} = \text{-1} \neq 0$$

Therefore, conditions of lemma (4) are satisfied, and system (2.3) has at least one $\omega - periodic$ solution in $DomL \cap \bar{\Omega}$.

4.2 Corollary 1

The set $K = (u(\tau), v(\tau), w(\tau)) : 0 \le u(\tau) \le M_1, 0 \le v(\tau) \le M_2, 0 \le w(\tau) \le M_3$ is the absorbing region of phase flows of dynamical system (2.3) in Ω.

5 Uniqueness and Global Attractivity of Periodic Solution

5.1 Proposition 2

Assume the perturbation parameters of dynamical system (2.3) are positive periodic functions, then the dynamical system(2.3) has a unique positive periodic solution, and globally attractive in absorbing region K.

Proof: Let $\Phi(\tau) = u(\tau), v(\tau), w(\tau))^T$ be a positive periodic solution of of system (2.3) and let $\Psi(\tau) = (u^*(\tau), y^*(\tau), z^*(\tau))^T$ be any solution of system (2.3) in K. We construct a positive definite Lyapunov's functional $F \in C[\mathbb{R}^3 \times \mathbb{R}^+, \mathbb{R}^+]$ defined as;

$$F(\tau) =\mid Inx(\tau) - Inx^*(\tau) \mid + \mid Iny(\tau) - Iny^*(\tau) \mid + \mid Inz(\tau) - Inz^*(\tau)|.$$

Using notations in [17] for upper right-derivative of the Lyapunov's functional and differentiating along the direction of trajectories of system (2.3) yields,

$$D^+F(\tau) = \left\{ \frac{\dot{x}(\tau)}{x(\tau)} - \frac{\dot{x}^*(\tau)}{x^*(\tau)} \right\} sign \mid x(\tau) - \dot{x}(\tau) \mid + \left\{ \frac{\dot{y}(\tau)}{y(\tau)} - \frac{\dot{y}^*(\tau)}{y^*(\tau)} \right\} sign \mid y(\tau) - \dot{y}(\tau) \mid + \left\{ \frac{\dot{z}(\tau)}{z(\tau)} - \frac{\dot{z}^*(\tau)}{z^*(\tau)} \right\} sign \mid z(\tau) - \dot{z}(\tau) \mid$$

$$\le \eta_1 \mid x(\tau) - x^*(\tau) \mid + \eta_2 \mid y(\tau) - y^*(\tau) \mid + \eta_3 \mid z(\tau) - z^*(\tau) \mid$$

where

$$\eta_1 = \frac{\eta^m M_2 + \beta^M}{(1 + M_1^l)^2} - \frac{\alpha^l}{\kappa^M} > 0, \frac{\beta^m + \sigma^m M_3}{(1 + M_2^l)^2} - \frac{\eta^l + \eta^l M_1^l}{(1 + M_1)^2} > 0,$$
$$\eta_3 = \frac{\sigma^m}{(1 + M_1)^2} - \frac{(1 + M_2)^2 + \sigma^l M_2^l (1 + M_1)}{(1 + M_1)(1 + M_2)^2} > 0$$

Choose $\delta = min(\eta_1, \eta_2, \eta_3) > 0$ and integrating both sides yields,

$$F(\tau) \le \delta \int_0^\tau (\mid x(\tau) - x^*(\tau) \mid + \mid y(\tau) - y^*(\tau) \mid + \mid z(\tau) - z^*(\tau) \mid) d\tau + F(0) < +\infty \tag{5.1}$$

The inequality (5.1) guarantees boundedness of the Lyapunov's functional on $[0, +\infty)$, and $(\mid x(\tau) - x^*(\tau) \mid + \mid y(\tau) - y^*(\tau) \mid + \mid z(\tau) - z^*(\tau) \mid) \in L^1(0, +\infty)$. Now, applying Barbalat's lemma [18], then $(\mid x(\tau) - x^*(\tau) \mid + \mid y(\tau) - y^*(\tau) \mid + \mid z(\tau) - z^*(\tau) \mid)$ is uniformly continuous on $[0, +\infty)$ and

$$\mid x(\tau) - x^*(\tau) \mid \to 0, \mid y(\tau) - y^*(\tau) \mid \to 0, \mid z(\tau) - z^*(\tau) \mid \to 0 \text{ as } \tau \to +\infty.$$

Therefore, system (2.3) assumed a unique globally attractive positive periodic solution, and trapped in the absorbing region K .

6 Application and Numerical Simulations

Consider the $\pi-$periodic coefficients of system (2.3) say, $\alpha(\tau) = 4.7688 + sin2\tau, \kappa(\tau) = 2.0064 + sin2\tau, \varepsilon(\tau) = 1.1249 + sin2\tau, \beta(\tau) = 0.543 + 0.2431sin2\tau, \xi = 0.041, \mu = 0.3804, \sigma = 1.0755, \mu = 0.1673, \alpha^l = 3.7688, \alpha^m = 5.7688, \beta^l = 0.2999, \beta^m = 0.7861, \kappa^l = 1.0044, \kappa^m = 3.0064, \varepsilon^l = 0.1249, \varepsilon^m = 2.1249, M_1^l = 0.5231, M_2^l = 0.3730, M_3^l = 0.5231, M_1 = 16.9816, M_2 = 17.6788, M_3 = 3.1951, \eta_1 = 0.3602, \eta_2 = 2.2390, \eta_3 = 0.5137, \delta = 0.3602$, subject to initial conditions, $x(0) = 1.0678, y(0) = 1.3730, z(0) = 0.6383$. It is easy to examine that the periodic coefficients satisfy boundedness conditions of proposition 1 and 2.

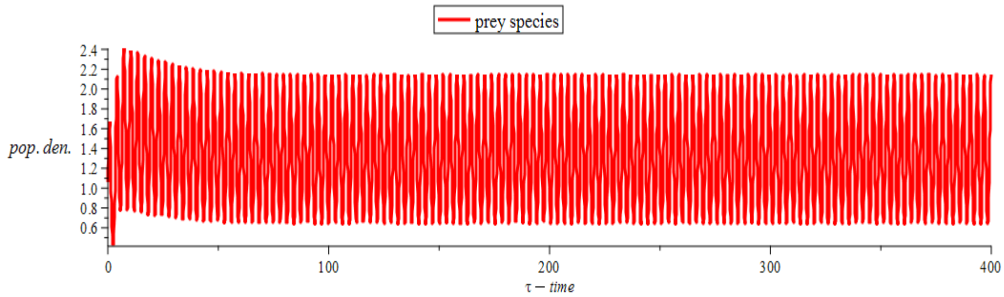

Fig. 1. Globally asymptotically stable periodic solution of prey species of system (2.3) at initial condition $x(0) = 1.0678, y(0) = 1.3730, z(0) = 0.6383$

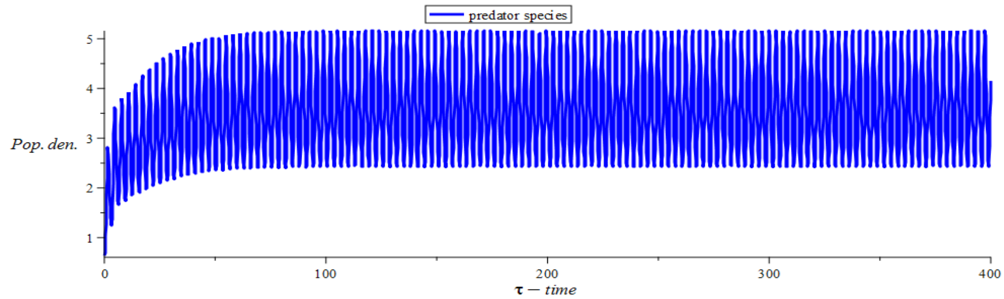

Fig. 2. Globally asymptotically stable periodic solution predator species of system (2.3) at initial condition $x(0) = 1.0678, y(0) = 1.3730, z(0) = 0.6383$

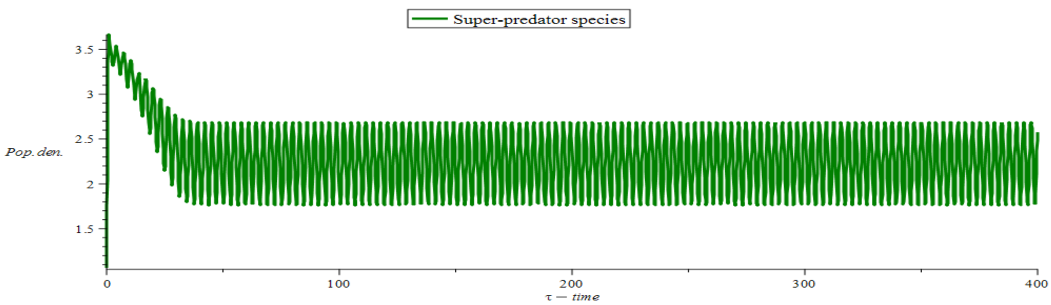

Fig. 3. Globally asymptotically stable periodic solutionsuper-predator species of system (2.3) at initial condition $x(0) = 1.0678, y(0) = 1.3730, z(0) = 0.6383$

7 Conclusion

This paper has established the necessary conditions for the existence of at least one positive periodic solution of an Extended Rosenzweig-MacArthur tri-trophic food chain model via Brouwers topological degree theory. Also, it has established the sufficient conditions for existence of a unique positive periodic solution of the model using Barbalats lemma and Lyapunovs functional. Consequently, the periodic solution is globally attractive in its invariance region. Thus, this model predicts and depicts a real-life ecological population dynamics as the perturbation parameters assumed periodic oscillations. Its connotes the natural ecological fluctuations.

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Competing Interests

Authors have declared that no competing interests exist.

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