Open Journal of Mathematical Analysis

Article

New inequalities based on harmonic log-convex functions

Imran Abbas Baloch[1,2,*] and Silvestru Sever Dragomir[3]

1 Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan.
2 Govt. College for Boys, Gulberg Higher Education Department, Punjab, Pakistan.
3 Mathematics, College of Engineering and Science, Victoria University, Melbourne City, Australia.; sever.dragomir@vu.edu.au
* Correspondence: iabbasbaloch@gmail.com

Received: 2 September 2019; Accepted: 22 December 2019; Published: 31 December 2019.

Abstract: Harmonic convexity is very important new class of non-convex functions, it gained prominence in the Theory of Inequalities and Applications as well as in the rest of Mathematics's branches. The harmonic convexity of a function is the basis for many inequalities in mathematics. Furthermore, harmonic convexity provides an analytic tool to estimate several known definite integrals like $\int_a^b \frac{e^x}{x^n}dx$, $\int_a^b e^{x^2}dx$, $\int_a^b \frac{\sin x}{x^n}dx$ and $\int_a^b \frac{\cos x}{x^n}dx\ \forall n \in \mathbb{N}$, where $a, b \in (0,\infty)$. In this article, some un-weighted inequalities of Hermite-Hadamard type for harmonic log-convex functions defined on real intervals are given.

Keywords: Harmonic convex functions, Hermite-Hadamard type inequalities, integral inequalities, harmonic log-convex functions.

MSC: 35B40, 35B41, 35B45, 35L05, 35R60, 58J37.

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1. Introduction

During the investigation of convexity, many researchers founded new classes of functions which are not convex in general. Some of them are the so called harmonic convex functions [1], harmonic (α, m)-convex functions [2], harmonic (s, m)-convex functions [3,4] and harmonic $(p,(s, m))$-convex functions [5]. For a quick glance on importance of these classes and applications, see [1–19] and references therein.

Definition 1. A function $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonic convex function on I if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le tf(y) + (1-t)f(x) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. If the inequality is reversed, then f is said to be harmonic concave.

In [5,20], Baloch *et al.* and Noor *et al.* also gave the definition of harmonic log-convex functions as follow:

Definition 2. A function $f : I \subseteq \mathbb{R}\backslash\{0\} \to (0,\infty)$ is said to be harmonic log-convex function on I if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le [f(x)]^{1-t}[f(y)]^{t} \tag{2}$$

holds for all $x, y \in I$ and $t \in [0,1]$. If the inequality is reversed, then f is said to be harmonic log-concave.

In [20], Noor *et al.* proved the following result for harmonic log-convex functions:

Theorem 3. *Let $I \subseteq \mathbb{R}\backslash\{0\}$ be an interval. If $f : I \to (0,\infty)$ is harmonic convex function, then*

$$f\left(\frac{2ab}{a+b}\right) \le \exp\left[\frac{ab}{b-a}\int_a^b \log\left(\frac{f(x)}{x^2}\right)dx\right] \le \sqrt{f(a)f(b)} \tag{3}$$

for all $a, b \in I$ and $a < b$.

Here, motivated by the above result we study the class of harmonic log-convex functions and present some new inequalities for this class of functions.

2. Main Results

The following result holds.

Theorem 4. *Let $f : I \subseteq \mathbb{R}\backslash\{0\} \to (0,\infty)$ be harmonic log-convex function. Then, for every $t \in [0,1]$, we have*

$$\begin{aligned}\int_a^b f(x)dx &\geq \int_a^b [f(x)]^{1-t}\left[\frac{a^2b^2}{[(a+b)x-ab]^2} f\left(\frac{abx}{(a+b)x-ab}\right)\right]^t dx \\ &\geq \begin{cases} (1-2t)a^2b^2 \int_{\frac{ab}{ta+(1-t)b}}^{\frac{ab}{(1-t)a+tb}} \frac{[(a+b)tu-ab]^{2(t-1)}}{[ab-(1-t)(a+b)u]^{2t}} f(u)du & \text{if } t \neq \frac{1}{2}; \\ \frac{2ab}{a+b}\ln(\frac{b}{a}) f\left(\frac{2ab}{a+b}\right) & \text{if } t = \frac{1}{2}. \end{cases}\end{aligned} \tag{4}$$

Proof. The cases $t = 0, \frac{1}{2}, 1$ are obvious. Assume that $t \in (0,1)\backslash\left\{\frac{1}{2}\right\}$. By the harmonic log-convexity of f we have

$$[f(x)]^{1-t}\left[f\left(\frac{abx}{(a+b)x-ab}\right)\right]^t \geq f\left(\frac{\frac{abx^2}{(a+b)x-ab}}{tx+(1-t)\frac{abx}{(a+b)x-ab}}\right) f\left(\frac{abx}{(a+b)tx-(2t-1)ab}\right) \tag{5}$$

for any $x \in [a,b]$. This allows that

$$[f(x)]^{1-t}\left[\frac{a^2b^2}{[(a+b)x-ab]^2} f\left(\frac{abx}{(a+b)x-ab}\right)\right]^t \geq \frac{a^{2t}b^{2t}}{[(a+b)x-ab]^{2t}} f\left(\frac{abx}{(a+b)tx-(2t-1)ab}\right). \tag{6}$$

Integrating the inequality (6) over x on $[a,b]$, we have

$$\int_a^b [f(x)]^{1-t}\left[\frac{a^2b^2}{[(a+b)x-ab]^2} f\left(\frac{abx}{(a+b)x-ab}\right)\right]^t dx \geq \int_a^b \frac{a^{2t}b^{2t}}{[(a+b)x-ab]^{2t}} f\left(\frac{abx}{(a+b)tx-(2t-1)ab}\right) dx.$$

Since $t \neq \frac{1}{2}$, then $u = \frac{abx}{(a+b)tx-(2t-1)ab}$ is the change of variable with $dx = \frac{(1-2t)a^2b^2}{[(a+b)tu-ab]^2}du$. For $x = a$, we get $u = \frac{ab}{ta+(1-t)b}$ and for $x = b$, we get $u = \frac{ab}{(1-t)a+tb}$. Therefore,

$$\int_a^b \frac{a^{2t}b^{2t}}{[(a+b)x-ab]^{2t}} f\left(\frac{abx}{(a+b)tx-(2t-1)ab}\right) dx = (1-2t)a^2b^2 \int_{\frac{ab}{ta+(1-t)b}}^{\frac{ab}{(1-t)a+tb}} \frac{[(a+b)tu-ab]^{2(t-1)}}{[ab-(1-t)(a+b)u]^{2t}} f(u)du,$$

and hence the second inequality (4) is proved. By the Hölder integral inequality for $p = \frac{1}{1-t}$, $q = \frac{1}{t}$, we have

$$\begin{aligned}&\int_a^b [f(x)]^{1-t}\left[\frac{a^2b^2}{[(a+b)x-ab]^2} f\left(\frac{abx}{(a+b)x-ab}\right)\right]^t dx \\ &\leq \left(\int_a^b \left([f(x)]^{1-t}\right)^{\frac{1}{1-t}} dx\right)^{1-t} \left(\int_a^b \left(\left[\frac{a^2b^2}{[(a+b)x-ab]^2} f\left(\frac{abx}{(a+b)x-ab}\right)\right]^t\right)^{\frac{1}{t}} dx\right)^t \\ &= \left(\int_a^b f(x)dx\right)^{1-t} \left(\int_a^b \frac{a^2b^2}{[(a+b)x-ab]^2} f\left(\frac{abx}{(a+b)x-ab}\right) dx\right)^t \\ &= \left(\int_a^b f(x)dx\right)^{1-t} \left(\int_a^b f(x)dx\right)^t = \int_a^b f(x)dx.\end{aligned}$$

This proves the first part of inequality (4). □

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: "The authors declare no conflict of interest."

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