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Degree affinity number of certain 2-regular graphs

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Abstract: This paper furthers the study on a new graph parameter called the degree affinity number. The degree affinity number of a graph G is obtained by iteratively constructing graphs, $G_1, G_2, \dots, G_k$ of increased size by adding a maximal number of edges between distinct pairs of distinct vertices of equal degree. Preliminary results for certain 2-regular graphs are presented.

Keywords: Degree affinity edge, degree affinity number.

MSC: 05C15, 05C38, 05C75, 05C85.

1. Introduction

It is assumed that the reader is familiar with the general notation and concepts in graphs. Good references are [1–3]. Throughout the study only finite, simple and undirected graphs will be considered. A paper which introduces the notion of the degree affinity number of a graph has been communicated, (see [4]).

It is known that a graph of order $n \geq 2$ has at least two vertices of equal degree. We recall that if two non-adjacent vertices $u, v \in V(G)$ with $deg_G(u) = deg_G(v)$ exist, then the added edge uv to obtain G' is called a *degree affinity edge*. For ease of reference we also recall the maximal degree affinity convention.

Maximal degree affinity convention (MDAC)

For a graph G the 1^{st}-iteration is the addition of degree affinity edges in respect of a maximal number of absolute distinct pairs of distinct non-adjacent vertices of equal degree, if such exist [4]. The graph obtained is labeled G_1. Hence, by the same convention it is possible to construct G_i from G_{i-1} provided that at least one (absolute distinct) pair of distinct non-adjacent vertices of equal degree exists in G_{i-1}. The MDAC terminates on the k^{th}-iteration if no further edges can be added.

We recall certain important results from [4].

Theorem 1. *[4] For an even cycle C_n, $n \geq 4$ the MDAC exhausts after $k = n-3$ iterations, $\eta(C_n)_{n,even} = \frac{n(n-3)}{2}$ and $G_{n-3} \cong K_n$.*

Corollary 1. *[4] For an odd cycle C_n, $n \geq 5$ the MDAC exhausts after $k = n-3$ iterations and $\eta(C_n)_{n,odd} = \frac{(n-2)(n-3)}{2}$.*

If a graph G has structural complexity, then finding $\eta(G)$ could be simplied by considering $\overline{G}$. However the dual problem must be considered. The dual to finding $\eta(G)$ is the deletion of the maximum number degree affinity edges from $\overline{G}$. The procedure is the iterative inverse of the MDAC and is denoted by, MDAC^{-1}. If a null graph (edgeless graph) results we say $\overline{G}$ reached *nullness*.

Theorem 2. *A graph G reaches completeness on exhaustion of the MDAC if and only if $\overline{G}$ reaches nullness on exhaustion of the MDAC^{-1}.*

Proof. If G reaches completeness on exhaustion of the MDAC then the set of degree affinity edges added is exactly, $E(\overline{G})$. By listing the degree affinity edges say s_i added to G per MDAC iteration $i = 1, 2, 3, \dots, k$ the inverse iterative deletion of degree affinity edge-lists s_j in $\overline{G}$ for $j = k, k-1, k-2, \dots, 1$, results in nullness.

The converse follows through similar reasoning. Therefore the result. □

In this paper we further the study in [4] for certain 2-regular graphs.

2. On regular graphs

To study the disjoint union of graphs, we distinguish between degree affinity edges *internal* to a graph G and those *external* to G. Let $V(G) = \{v_i : 1 \leq i \leq n\}$ and $V(H) = \{u_j : 1 \leq j \leq m\}$. In the disconnected graph $G \cup H$ and through all iterations of the MDAC applied thereto, degree affinity edges of the form v_iv_k or u_ju_t are called *internal* to G or H, respectively. Degree affinity edges of the form v_iu_j are called *external* to both G and H. Furthermore, if all vertex degrees $deg_G(v_i)$, $v_i \in V(G)$ are weighted by a constant $a \in \mathbb{N}$ we denote the graph with weighted degrees by, G^{+a}.

Lemma 1. *If graph G of order n has degree sequence $(deg_G(v_i) : deg_G(v_i) \geq deg_G(v_{i+1}), 1 \leq i \leq n-1)$ and G^{+a} has degree sequence $(deg_G(v_i) + a : deg_G(v_i) \geq deg_G(v_{i+1}), 1 \leq i \leq n-1)$ then the degree affinity properties of G^{+a} are identical to that of G.*

Proof. Since

(a) $v_iv_j \in E(G)$ and $deg_g(v_i) = deg_G(v_j)$ in $G \Leftrightarrow v_iv_j \in E(G^{+a})$ and $deg_g(v_i) + a = deg_G(v_j) + a$ in G^{+a} or;
(b) $v_iv_j \in E(G)$ and $deg_g(v_i) \neq deg_G(v_j)$ in $G \Leftrightarrow v_iv_j \in E(G^{+a})$ and $deg_g(v_i) + a \neq deg_G(v_j) + a$ in G^{+a} or;
(c) $v_iv_j \notin E(G)$ and $deg_g(v_i) = deg_G(v_j)$ in $G \Leftrightarrow v_iv_j \notin E(G^{+a})$ and $deg_g(v_i) + a = deg_G(v_j) + a$ in G^{+a} or;
(d) $v_iv_j \notin E(G)$ and $deg_g(v_i) \neq deg_G(v_j)$ in $G \Leftrightarrow v_iv_j \notin E(G^{+a})$ and $deg_g(v_i) + a \neq deg_G(v_j) + a$ in G^{+a},

the result follows immediately. □

An immediate consequence of Lemma 1 follows.

Theorem 3. *Let graphs G and H both be of order n and r-regular then, $\eta(G \cup H) = \eta(G) + \eta(H) + n^2$.*

Proof. Clearly, the disjoint union $G \cup H$ is of order $2n$. Hence, for each of the initial n iterations there exist n distinct pairs of distinct vertices $\{u, v\}$, $u \in V(G)$, $v \in V(H)$ such that $uv \notin E(G \cup H)$ and $deg_{(G\cup H)_{i-1}}(u) = deg_{(G\cup H)_{i-1}}(v) = deg_G(u) + (i-1) = deg_H(v) + (i-1)$. Therefore, after the initial n iterations all the possible degree affinity edges between G and H have been added. Without loss of generality, the graph G can be viewed as a graph with weighted vertex degrees i.e., G^{+n}. By Lemma 1 the degree affinity properties of G^{+n} are identical to that of G, (similarly for H^{+n}). The MDAC is now simultaneously applied to G^{+n} and H^{+n}. Thus, $\eta(G \cup H) = \eta(G) + \eta(H) + n^2$. □

Applying Theorem 3 could present difficulty. It is easy to see that if both graphs G and H, independently reach completeness on exhaustion of the MDAC, the grouped iterations i.e., (a) first apply MDAC independently to G and H, (b) thereafter apply MDAC between $\mathbb{A}(G)$ and $\mathbb{A}(H)$ or interchanging the grouped iterations, yield the same result. This observations does not hold in general. Consider the two 3-regular graphs of order 6 (Figures 1 and 2).

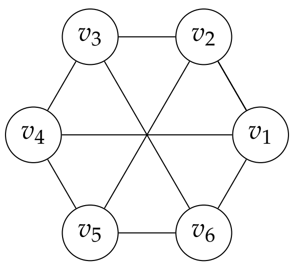

Figure 1. Graph G which up to isomorphism can be exhausted by adding degree affinity edges say, v_2v_6, v_3v_5.

It follows from Figures 1 and 2 that applying the MDAC to G and H independently yields graphs $\mathbb{A}(G)$ with degree sequence $(4,4,4,4,3,3)$ and $\mathbb{A}(H)$ with degree sequence $(5,5,5,5,5,5)$. Hence, $\mathbb{A}(G)$ and $\mathbb{A}(H)$ remain disjoint in $\mathbb{A}(G) \cup \mathbb{A}(H)$. The second approach, to first add degree affinity edges v_iu_j, $1 \leq i,j \leq 6$

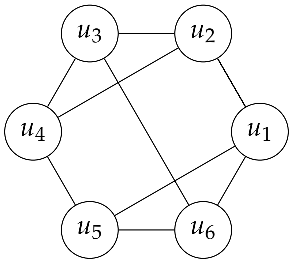

Figure 2. Graph H reaches completeness in two iterations by adding the degree affinity edges say, (i) u_1u_4, u_2u_6, u_3u_5 then, (ii) u_1u_3, u_2u_5, u_4u_6.

(external to both G and H) and then adding the degree affinity edges v_iv_j and u_iu_j, $i \neq j$, $1 \leq i, j \leq 6$ to G^{+6} and H^{+6} yields the maximum number of degree affinity edges.

2.1. Disjoint union of cycles

Recall that a cycle on $n \geq 3$ vertices is a graph denoted by, C_n and $V(C_n) = \{v_i : 1 \leq i \leq n\}$, $E(C_n) = \{v_iv_i + 1 : 1 \leq i \leq n-1\} \cup \{v_nv_1\}$. The family of 2-regular graphs are all graphs such that, each graph G consists of one or more (disconnected or disjoint union of) cycles. The numbers a_n, $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$ of 2-regular graphs on n vertices are given by $0, 0, 1, 1, 1, 2, 2, 3, 4, 5, \ldots, p(t) - p(t-1) - p(t-2) + p(t-3), \ldots$ with $p(t)$ the partition function, (see https://oeis.org/A008483).

The results for both even and odd cycles are provided by Theorem 1 and Corollary 1. Categories of disconnected of 2-regular graphs (disjoint union of cycles) of order $3 \leq n \leq 10$ will be discussed.

Clearly, for $n = 3, 4, 5$ there exists a unique 2-regular graph each i.e. C_3, C_4, C_5. The other categories are;

(i) $C_3 \cup C_3$,
(ii) $C_3 \cup C_4$,
(iii) $C_3 \cup C_5$ and $C_4 \cup C_4$,
(iv) $C_3 \cup C_6$, $C_4 \cup C_5$ and $C_3 \cup C_3 \cup C_3$ (or $3C_3$),
(v) $C_3 \cup C_7$, $C_4 \cup C_6$, $C_5 \cup C_5$ (or $2C_5$) and $2C_3 \cup C_4$.

Theorem 3 read together with Theorem 1 leads to a proposition which requires no further proof.

Proposition 1. *(a) For the disjoint union of two copies of an even cycle C_n, $n \geq 4$ the MDAC exhausts after $k = 2n-3$ iterations, $\eta(C_n \cup C_n)_{n,even} = n(n-3)$ and $\mathbb{A}(C_n \cup C_n) \cong K_{2n}$.*
(b) For the disjoint union of two copies of an odd cycle C_n, $n \geq 3$ the MDAC exhausts after $k = 2n-3$ iterations, $\eta(C_n \cup C_n)_{n,odd} = 2n^2 - 5n + 6$.

Definition 1. For a graph G of order n and $1 \leq t \leq n$, select $X \subseteq V(G)$, $|X| = t$ such that a minimum number of vertices $v \in X$ are adjacent. The set X is said to be an optimal near-independent selection.

For $X \subseteq V(G)$ the subgraph induced by X is denoted by, $\langle X \rangle$. Definition 1 can be put differently i.e. select $X \subseteq V(G)$, $|X| = t$ such that $\langle X \rangle$ has a minimum number of edges.

Proposition 2. *(i) $\eta(C_3 \cup C_3) = 9$.*
(ii) $\eta(C_3 \cup C_4) = 10$.
(iii) (a). $\eta(C_3 \cup C_5) = 11$.
(b). $\eta(C_4 \cup C_4) = 20$.
(iv) (a). $\eta(C_3 \cup C_6) = 11$.
(b). $\eta(C_4 \cup C_5) = 21$.
(c). $\eta(C_3 \cup C_3 \cup C_3) \geq 19$.
(v) (a). $\eta(C_3 \cup C_7) = 19$.
(b). $\eta(C_4 \cup C_6) = 24$.

(c). $\eta(C_5 \cup C_5) = 31$.

(d). $\eta(2C_3 \cup C_4) \geq 21$.

Proof. (i) Follows from Proposition 1.

(ii) Let C_3 be on vertices v_1, v_2, v_3 and C_4 on vertices u_1, u_2, u_3, u_4. Without loss of generality add the external degree affinity edges between v_1, v_2, v_3 and u_1, u_2, u_3 in 3 iterations. Now vertices u_1, u_2, u_3 each, has degree of 5. Add degree affinity edge u_1u_3 to exhaust the MDAC. Clearly, $\eta(C_3 \cup C_4) = 10$.

(iii) (a). Let C_3 be on vertices v_1, v_2, v_3 and C_5 on vertices u_1, u_2, u_3, u_4, u_5. Note that if degree affinity edges between v_1, v_2, v_3 and u_1, u_2, u_3 are added then u_4, u_5 remain adjacent. An *optimal near-independent* selection will, without loss of generality be say, vertices u_1, u_2, u_4. Therefore, in 1^{st}-iteration add the degree affinity edges u_3u_5, v_1u_1, v_2u_2, v_3u_4. During the 2^{nd}- and 3^{rd}-iteration reach completion between v_1, v_2, v_3 and u_1, u_2, u_4. In the exhaustive 4^{th}-iteration add either u_1u_4 or u_2u_4. Clearly, $\eta(C_3 \cup C_5) = 11$.

(b). Follows from Proposition 1.

(iv) (a). Let C_3 be on vertices v_1, v_2, v_3 and C_6 on vertices $u_1, u_2, u_3, u_4, u_5, u_6$. Without loss of generality an *optimal near-independent* selection of vertices in C_6 will be say, u_1, u_3, u_5. Hence, in three iterations add the degree affinity edges by first adding , either u_2u_4 or u_2u_6 or u_4u_6 together with v_1u_1, v_2u_2, v_3u_3. Thereafter complete the degree affinity edges between C_3 and C_6. Finally, add either u_1u_3 or u_1u_5 or u_3u_5. Clearly, $\eta(C_3 \cup C_6) = 11$.

(b). The result follows through similar reasoning in the proof of (ii).

(c). Let the vertices of three copies of C_3 be labeled v_1, v_2, v_3 and u_1, u_2, u_3 and w_1, w_2, w_3, respectively. For the convenience of reference label the cycles, G_1, G_2, G_3 respectively.

Case 1: Add all the degree affinity edges between any pair of C_3 cycles. Since the MDAC is exhausted, $\eta(C_3 \cup C_3 \cup C_3) \geq 9$.

Case 2: In the 1^{st}-iteration consider pairs of cycles in the order, (G_1, G_2), (G_2, G_3), (G_3, G_1), (G_1, G_2) and add the degree affinity edges say, v_1u_1, u_2w_2, w_3v_3, v_2u_3. In the 2^{nd}-iteration consider pairs of cycles in the order, (G_2, G_3), (G_3, G_1), (G_1, G_2) and add the degree affinity edges say, u_1w_2, w_3v_1, v_2u_2, v_3u_3. In the 3^{rd}-iteration consider pairs of cycles in the order, (G_3, G_1), (G_1, G_2) and add the degree affinity edges say, w_2v_3, w_3u_3, v_1u_2, v_2u_1. In the 4^{th}-iteration consider pairs of cycles in the order, (G_1, G_2), (G_2, G_3), (G_3, G_1) and add the degree affinity edges say, v_1u_3, v_3u_1, u_2w_3, w_2v_2. Finally to exhaust the MADC add degree affinity edges, v_2w_3, v_1w_2, v_3u_2. Since the methodology has only been tested exhaustively and not proven to yield the maximum number of degree affinity edges the best result is, $\eta(C_3 \cup C_3 \cup C_3) \geq max\{9, 19\} = 19$.

(v) (a). **Case 1:** Clearly since C_3 is complete, $\eta(C_3 \cup C_7) \geq 10 = \eta(C_7)$.

Case 2: Let C_3 be on vertices v_1, v_2, v_3 and C_7 on vertices $u_1, u_2, u_3, u_4, u_5, u_6, u_7$. It follows easily that if the MDAC is applied between C_3 and vertices u_1, u_2, u_3 then, $\eta(C_3 \cup C_7) \geq 13$.

Case 3: Apply the MDAC between C_3 and vertices u_1, u_2, u_4 as follows. Add degree affinity edges v_1u_1, v_2u_2, v_3u_4 as well as say, u_3u_6, u_5u_7. In the 2^{nd}-iteration add, v_1u_2, v_2u_4, v_3u_1 as well as say, u_3u_5. In the 3^{rd}-iteration add v_1u_4, v_2u_1, v_3u_2. Exhaust the MDAC in the 4^{th}-iteration by adding say, u_1u_4. Hence, $\eta(C_3 \cup C_7) \geq 13$.

Case 4: Apply the MDAC between C_3 and vertices u_1, u_2, u_5. Through similar reasoning as in Case 3 it follows that, $\eta(C_3 \cup C_7) \geq 14$.

Case 5: Without loss of generality an *optimal near-independent* selection of vertices in C_7 will be say, u_1, u_3, u_5. In the 1^{st}-iteration add the degree affinity edges, v_1u_1, v_2u_3, v_3u_5 and u_2u_7, u_4u_6. In the 2^{nd}-iteration ad the degree affinity edges v_1u_3, v_3u_1, v_2u_5, u_2u_6, u_4u_7. In the 3^{rd}-iteration ad the degree affinity edges v_1u_5, v_2u_1, v_3u_3, u_2u_4. In the 3^{rd}-iteration ad the degree affinity edges v_1u_2, v_3u_4, u_1u_5. Finally, add v_1u_4 and u_2u_5. Since all possibilities up to isomorphisms have been considered the result is, $\eta(C_3 \cup C_7) = \max\{10, 13(repeated), 14, 19\} = 19$.

(b) Let C_4 be on vertices v_1, v_2, v_3, v_4 and C_6 on vertices $u_1, u_2, u_3, u_4, u_5, u_6$.

Case 1: If the MDAC is applied to C_4 and C_6 independently it follows that $\eta(C_4 \cup C_6) \geq 11$.

Case 2: Let C_4 be on vertices v_1, v_2, v_3, v_4 and C_6 on vertices $u_1, u_2, u_3, u_4, u_5, u_6$. By applying the MDAC between C_4 and vertices u_1, u_2, u_3, u_4 it follows easily that exhaustion is reached between six iterations. Hence, $\eta(C_4 \cup C_6) \geq 21$.

Case 3: By applying the MDAC between C_4 and vertices u_1, u_2, u_3, u_5 it follows easily that exhaustion is reached between six iterations. Note that in the 1^{st}-iteration the degree affinity edge u_4u_6 was added thus, $\eta(C_4 \cup C_6) \geq 22$.

Case 4: By applying the MDAC between C_4 and vertices u_1, u_2, u_4, u_5 it follows easily that exhaustion is reached between six iterations. Note that in the 1^{st}-iteration the degree affinity edge u_3u_6 was added thus, $\eta(C_4 \cup C_6) \geq 23$. Since all possibilities up to isomorphisms have been considered the result is, $\eta(C_4 \cup C_6) = \max\{11, 21, 22, 23\} = 24$.

(c). Follows from Proposition 1.

(d). Through similar reasoning as that in (iv)(c) the result is, $\eta(2C_3 \cup C_4) \geq 21$.

□

Another approach to find the results (iv)(c) and (v)(a)-(d) is proposed. Consider (iv)(c). In the first iteration add the degree affinity edges say, v_1u_1, u_2w_2, w_3v_3 and v_2u_3. Relabel the vertices as follows: $v_1 \mapsto z_1$, $u_1 \mapsto z_2$, $u_3 \mapsto z_3$, $u_2 \mapsto z_4$, $w_2 \mapsto z_5$, $w_1 \mapsto z_6$, $w_3 \mapsto z_7$, $v_3 \mapsto z_8$, $v_2 \mapsto z_9$. A chorded cycle as depicted in Figure 3 is obtained.

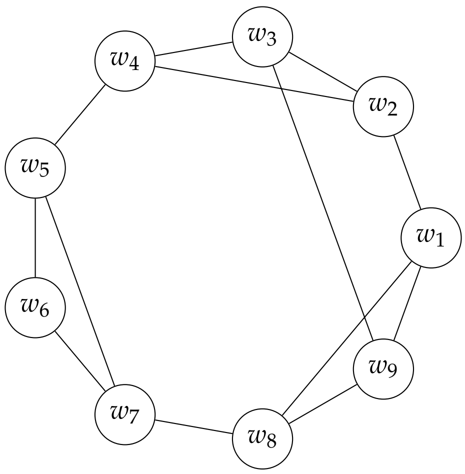

Figure 3. Chorded cycle.

The avenue for researching chorded cycles could lead to an improved methodology.

2.2. Chorded cycles

A chorded cycle with ℓ chords is denoted by $C_n^{\sim\ell}$, $1 \leq \ell \leq \frac{n(n-3)}{2}$. The observations from Figures 1 and 2 read together with Theorem 1 and Corollary 1 provide a proposition which requires no further proof.

Proposition 3. *For a cycle C_n,*

(a) $\eta(C_n^{\sim\ell}) \leq \frac{n(n-3)}{2} - \ell$ *if n is even.*
(b) $\eta(C_n^{\sim\ell}) \leq \frac{(n-2)(n-3)}{2} - \ell$ *if n is odd.*

Chorded cycles with ℓ independent chords, i.e., no pair of distinct chords share an end-vertex will be denoted by $C_n^{\sim\ell(i)}$, $1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor$. A researcher, Dillon Lareau investigated a problem which is described as, finding "the number of ways of dividing n labeled items into k unlabeled boxes as evenly as possible". In the graph coloring context the Lareau problem can be stated as, finding "the number of ways of coloring n labeled and isolated vertices (or the labeled vertices of the null graph, $\mathfrak{N}_n$) with k distinct colors as evenly as possible". The aforesaid problem was investigated in the context of chromatic completion of graphs. For an introduction to chromatic completion of a graph G, see [5–9]. In the aforesaid context the number of ways of coloring was called the lucky number denoted by, $L(n, k)$. The vertex set partitions which correspond to the "ways of coloring" are called lucky partitions. A closed formula was announce by Dillon Lareau (11 June 2019) which is given by, $L(n,k) = \frac{n!}{A!B!(\lceil \frac{n}{k} \rceil !)^A (\lfloor \frac{n}{k} \rfloor !)^B}$, $A = n(mod\ k)$, $B = k - A$, (see https://oeis.org/A308624). Let the number of ways ℓ independent chords can be added to C_n be denoted by, $\bigoplus_\ell(C_n)$.

Determining $\bigoplus_\ell(C_n)$ presents difficulty because no computer algorithm is available to generate the required partitions. To illustrate this difficulty consider finding all 2-chorded cycles of order 6. To begin, first find the corresponding $L(6,(6-2)) = \frac{6!}{2!2!(2!)^2(1!)^2} = 45$ lucky partitions.

The corresponding lucky partitions of $V(\mathfrak{N}_6)$ (null graph order 6) are;

$$\begin{gathered}
\{\{v_1,v_2\},\{v_3,v_4\},\{v_5\},\{v_6\}\},\\
\{\{v_1,v_2\},\{v_3,v_5\},\{v_4\},\{v_6\}\},\\
\{\{v_1,v_2\},\{v_3\},\{v_4,v_5\},\{v_6\}\},\\
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\{\{v_1,v_5\},\{v_2\},\{v_3\},\{v_4,v_6\}\},\\
\{\{v_1,v_6\},\{v_2,v_5\},\{v_3\},\{v_4\}\},\\
\{\{v_1\},\{v_2,v_5\},\{v_3,v_6\},\{v_4\}\},\\
\{\{v_1\},\{v_2,v_5\},\{v_3\},\{v_4,v_6\}\},\\
\{\{v_1,v_6\},\{v_2\},\{v_3,v_5\},\{v_4\}\},\\
\{\{v_1\},\{v_2,v_6\},\{v_3,v_5\},\{v_4\}\},\\
\{\{v_1\},\{v_2\},\{v_3,v_5\},\{v_4,v_6\}\},\\
\{\{v_1,v_6\},\{v_2\},\{v_3\},\{v_4,v_5\}\},\\
\{\{v_1\},\{v_2,v_6\},\{v_3\},\{v_4,v_5\}\},\\
\{\{v_1\},\{v_2\},\{v_3,v_6\},\{v_4,v_5\}\}.
\end{gathered}$$

Finally, eliminate all lucky partitions which have some 2-element subset which is an edge of the cycle C_6. This yields,

$$\begin{aligned}
&\{\{v_1,v_3\},\{v_2,v_4\},\{v_5\},\{v_6\}\},\\
&\{\{v_1,v_3\},\{v_2,v_5\},\{v_4\},\{v_6\}\},\\
&\{\{v_1,v_3\},\{v_2,v_6\},\{v_4\},\{v_5\}\},\\
&\{\{v_1,v_3\},\{v_2\},\{v_4,v_6\},\{v_5\}\},\\
&\{\{v_1,v_4\},\{v_2,v_5\},\{v_3\},\{v_6\}\},\\
&\{\{v_1,v_4\},\{v_2\},\{v_3,v_5\},\{v_6\}\},\\
&\{\{v_1,v_4\},\{v_2,v_6\},\{v_3\},\{v_5\}\},\\
&\{\{v_1,v_4\},\{v_2\},\{v_3,v_6\},\{v_5\}\},\\
&\{\{v_1,v_5\},\{v_2,v_4\},\{v_3\},\{v_6\}\},\\
&\{\{v_1\},\{v_2,v_4\},\{v_3,v_5\},\{v_6\}\},\\
&\{\{v_1\},\{v_2,v_4\},\{v_3,v_6\},\{v_5\}\},\\
&\{\{v_1,v_5\},\{v_2,v_6\},\{v_3\},\{v_4\}\},\\
&\{\{v_1,v_5\},\{v_2\},\{v_3,v_6\},\{v_4\}\},\\
&\{\{v_1,v_5\},\{v_2\},\{v_3\},\{v_4,v_6\}\},\\
&\{\{v_1\},\{v_2,v_5\},\{v_3,v_6\},\{v_4\}\},\\
&\{\{v_1\},\{v_2,v_5\},\{v_3\},\{v_4,v_6\}\},\\
&\{\{v_1\},\{v_2,v_6\},\{v_3,v_5\},\{v_4\}\},\\
&\{\{v_1\},\{v_2\},\{v_3,v_5\},\{v_4,v_6\}\}.
\end{aligned}$$

Clearly, $\bigoplus_2(C_6) = 18$.

Corollary 2. *For a cycle C_n, $n \geq 3$, we have the inequality, $\bigoplus_\ell(C_n) < L(n, n-\ell)$.*

Proof. Since some lucky partitions have at least one 2-element subset which is an edge of the cycle C_n, the result holds. □

Let a family of non-isomorphic independent ℓ-chorded cycles be denoted by, $\mathfrak{C}_n^{\sim\ell(i)}$. For the example above and without loss of generality it follows that,

$\mathfrak{C}_6^{\sim2(i)} = \{\{\{v_1,v_3\},\{v_2,v_4\},\{v_5\},\{v_6\}\},\{\{v_1,v_3\},\{v_2,v_5\},\{v_4\},\{v_6\}\},\{\{v_1,v_3\},\{v_2\},\{v_4,v_6\},\{v_5\}\},$ $\{\{v_1,v_4\},\{v_2,v_5\},\{v_3\},\{v_6\}\}\}$.

We now open the avenue for researching the degree affinity properties of $C_n \cup C_m$, $m > n \geq 3$. If all external degree affinity edges between C_n and C_m as well as the required internal degree affinity edges to C_m is added during the 1^{st}-iteration of the MDAC, a Hamiltonian graph is obtained. This Hamiltonian graph is a chorded cycle, $C_{n+m}^{\sim\ell(i)}$, $\ell = n + \lfloor\frac{m-n}{2}\rfloor \geq 3$. The Hamilton cycle is not unique because the choice of connecting pairs of vertices between C_n and C_m is not unique. By exhausting the ways in which the pairs of vertices can be connected and by exhausting the Hamilton cycles within each chorded cycle, a family $\mathfrak{C}_{n+m}^{\sim\ell(i)}$ can be generated. Therefore, from Definition 2.1 in [4] it follows that, for $m \geq n$, $\eta(C_n \cup C_m) \leq (n + \lfloor\frac{m-n}{2}\rfloor) + max\{\eta(C_{n+m}^{\sim\ell(i)}) : C_{n+m}^{\sim\ell(i)} \in \mathfrak{C}_{n+m}^{\sim\ell(i)}, \ell = n + \lfloor\frac{m-n}{2}\rfloor\}$.

Example 1. Consider $C_3 \cup C_3$. Let the cycles be on vertices v_1, v_2, v_3 and u_1u_2, u_3 respectively. Without loss of generality add the three external degree affinity edges, v_1u_1, v_2u_2, v_3u_3. Relabel the vertices as follows: $v_1 \mapsto z_1$, $v_3 \mapsto z_2$, $v_2 \mapsto z_3$, $u_2 \mapsto z_4$, $u_3 \mapsto z_5$, $u_1 \mapsto z_6$.

It implies that $\eta(C_3 \cup C_3) \leq 3 + max\{2, 6\} = 9$. Note that since the $C_3^{'s}$ are complete, a unique independent chorded cycle $C_6^{\sim3(i)}$ is obtained which yields equality. Hence, $\eta(C_3 \cup C_3) = 9$.

3. Conclusion

Besides doing "mathematics for the sake of mathematics", motivation related to applications of the notion of degree affinity has been stated in [4]. Current research into an application related to chemical affinity between atoms or molecular affinity in molecular structures is underway. It is hoped to report on results soon.

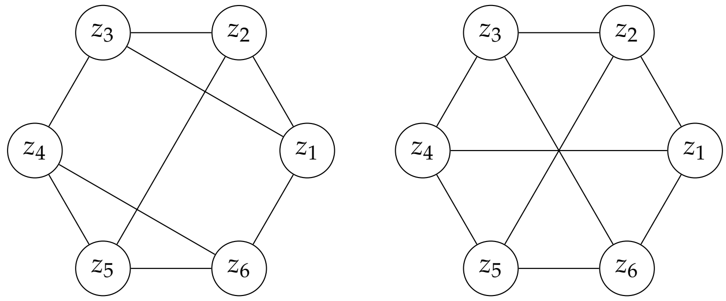

Figure 4. $\mathfrak{C}_6^{\sim 3(i)}$ has two distinct $C_6^{\sim 3(i)}$.

Investigating the new parameter $\eta(G)$ for the disjoint union of cycles poses numerous challenges. It is the considered view of the author that for labeled graphs, the development of a computer generator of lucky partitions followed by the reduction of the partitions to the permissible partitions is key to furthering this research meaningfully. If a methodology can be developed to generate a family of non-isomorphic ℓ-chorded cycles it will be worthy to further results.

Finding a closed formula for $\bigoplus_\ell(C_n)$, $n \geq 3$ is worthy of endeavour. Finding improvement on the inequality of Corollary 2 remains open.

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